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PARTIAL BOOLEAN σ-ALGEBRAS

The present note contains some simple generalizations of notions introduced in [1] and [2]. We refer the reader to those papers for all relevant definitions.

DEFINITION 1. We shall say that the system $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1 \rangle$ is a partial Boolean σ-algebra ($PB\sigma$Algebra) if $\underline{B}$ is a partial Boolean algebra and for every denumerable sequence $\{a_n\}_{n \in N}$ of mutually commeasurable elements of $B(a_m \overset{|}{\circ} a_n$ for $m, n \in N)$ there exists $b \in B$ that

(i) $\forall_{n \in N} a_n \subseteq b$ (i.e., $a_n \overset{|}{\circ} b$ and $a_n \vee b = b$)

(ii) $\forall_{c \in B} \forall_{n \in N} (a_n \overset{|}{\circ} c \Rightarrow b \overset{|}{\circ} c)$

(iii) $\forall_{c \in B} \forall_{n \in N} (a_n \subseteq c \Rightarrow b \subseteq c)$

We shall write: $b = \bigvee\limits_{n=1}^{\infty} a_n$. $\bigvee$ is a well-defined partial, infinite function.

EXAMPLE 1. Let $L(H)$ be the set of all closed subspaces of a separable Hilbert space. It is well-known that $\underline{L}(H) = \langle L(H); \overset{|}{\circ}; \vee; \neg; H \rangle$ is a PAAlgebra, where $\overset{|}{\circ}$ is defined as follows:

$H_1 \overset{|}{\circ} H_2$ iff there exist a basis ε and subsets
$\varepsilon_i \subseteq \varepsilon (i = 1, 2)$ such that ε_i is a basis in $H_i (i = 1, 2)$
and $\neg H_0$ is the orthogonal complementation of H_0 ([1]).
$\underline{L}(H)$ is also a $PB\sigma$Algebra because of the following simple fact:

If $H_1, H_2, \ldots, H_n, \ldots$ are mutually commeasurable $(H_m \overset{|}{\circ} H_n$
for $m, n \in N)$ then there exists a basis ε in H and subsets
$\varepsilon_n \subseteq \varepsilon (n = 1, 2, \ldots)$ such that ε_n is a basis in $H_n (n = 1, 2, \ldots)$

As before one can consider the logic based on $PB\sigma$Algebras. Strictly speaking, let $P_0 = \{x_\nu : \nu < \omega_1\}$ be the set of variables. P – the set of formulas of infinite length-is defined as the least set satisfying the conditions:

(1) $P_0 \subseteq P$
(2) If $\varphi_1, \varphi_2, \ldots \in P$, then $\bigvee\{\varphi_n : n \in N\} \in P$
(3) If $\varphi \in P$, then $\neg\varphi \in P$.

By a valuation h of $\underline{P}$ in $\underline{B}$ ($\underline{B} \in PB\sigma A$) we shall mean every partial homomorphism of $\underline{P}$ into $\underline{B}$, i.e., a function whose domain $Dom(h) \subseteq P$ ($h : Dom(h) \to B$) is the least set, such that:

(a) Some variables belong to $Dom(h)$ (but not necessarily all); if $x_\nu \in Dom(h)$, then $hx_\nu \in B$.
(b) If $\varphi_1, \varphi_2, \ldots \in Dom(h)$ and $h\varphi_1, h\varphi_2, \ldots$ are mutually comeasurable, then $\bigvee\{\varphi_n : n \in N\} \in Dom(h); h(\bigvee\{\varphi_n : n \in N\}) = \bigvee\limits_{n=1}^{\infty} h\varphi_n$.
(c) If $\varphi \in Dom(h)$, then $\neg\varphi \in Dom(h); h\neg\varphi = \neg(h\varphi)$.

By a model $\mathcal{M}$ we shall mean any pair $\mathcal{M} = \langle\underline{B}, h\rangle$, where $\underline{B} \in PB\sigma A$ and h is a valuation ($h : \underline{P} \to \underline{B}$). We shall say that φ is valid in $\mathcal{M} = \langle\underline{B}, h\rangle$ if $\varphi \in Dom(h)$ and $h\varphi = 1$. A formula φ holds in $\underline{B}$ ($\underline{B} \in PB\sigma A$) if $h\varphi = 1$ for every valuation $h : \underline{P} \to \underline{B}$, such that $\varphi \in Dom(h)$. Similarly, we shall say that an identity $\varphi = \phi$ holds in $\underline{B}$ ($\varphi, \phi \in P$) if $h\varphi = j\phi$ for every valuation h ($h : \underline{P} \to \underline{B}$), such that $\varphi, \phi \in Dom(h)$. A formula φ is valid if it holds in every $PB\sigma$Algebra.

It is easy now to define a consequence operation Cn on $\underline{P}$. Let $\alpha \in P$ and $X \subseteq P$. Then $\alpha \in Cn(X)$ iff for every $PB\sigma$Algebra $\underline{B}$ and every valuation $h : \underline{P} \to \underline{B}$ it holds:

$$\forall_{\beta \in X}(\beta \in Dom(h) \Rightarrow h\beta = 1) \Rightarrow (\alpha \in Dom(h) \Rightarrow h\alpha = 1).$$

Let us observe that $Cn(\emptyset)$ is the set of all valid formulas. In a similar way one can construct the logic based on PBAlgebras. Then Cn is a structural consequence operation.

As in the case of PBAlgebras it is easy to axiomatize $Cn(\emptyset)$ by a simple modification of methods of Kochen and Specker [1].

It is interesting to establish connections between $PB\sigma$Algebras and quantum logic (i.e., weakly modular orthoposets). Let us recall this notion:

$\underline{L} = \langle L; ^\perp; \subseteq; 1 \rangle$ is a quantum Logic if it is a finitely additive quantum logic (see [2], p. 168) and satisfies the condition:

$(L1)^*$ For every denumerable sequence $\{a_n\}_{n \in N}$ of mutually
orthogonal elements of L there exists the least upper
bound $\bigvee\limits_{n=1}^{\infty} a_n$.

A partial Boolean σ-algebra is transitive if it is transitive as PBAlgebra (see [2], p.167).

THEOREM 1.   *(a) Let $\underline{B} = \langle B; \overset{|}{\circ}; \vee; \neg; 1 \rangle$ be a transitive partial Boolean σ-algebra. Then $\underline{B}^* = \langle B; \subseteq; \perp; 1 \rangle$, where $a^\perp = \neg a$, is a quantum Logic.*
(b) Let $\underline{L} = \langle L; \subseteq; \perp; 1 \rangle$ be a quantum logic. Then $L^0 = \langle ; \leftrightarrow; \vee, \neg; 1 \rangle$ is a transitive $PB\sigma$Algebra, where $\neg a = a^\perp$ and $a \vee b$ is the l.u.b. of a and b ($a \leftrightarrow b$).
(c) Transitive partial Boolean σ-algebras $\underline{B}$ and $\underline{B}^{*0}$ are identical.
(d) Quantum logics $\underline{L}$ and $\underline{L}^{0*}$ are identical.

Partial Boolean σ-algebras are generalizations of quantum logics (because there exist $PB\sigma$Algebras which are not inbeddable into any transitive $PB\sigma$Algebra). It would be interesting from physical point of view to develope a calculus of observables just in partial Boolean σ-algebras. Connections with some physical problems like Hidden-Variable Theories are easily seen in the following Imbedding Theorems:

THEOREM 2A.   *Let $\underline{L} \in PB\sigma A$. The following conditions are equivalent:*

  (i)   *There exists a homomorphism of $\underline{L}$ into a Boolean σ-algebra.*
  (ii) *No formula valid in all Boolean σ-algebras is refutable in $\underline{L}$.*
  (iii) *For every denumerable set $A = \{a_1, a_2, \ldots\} \subseteq L$ there exists a sequence $(\varepsilon_1, \varepsilon_2, \ldots)$, $\varepsilon_n = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right.$, such that for every set of mutually commeasurable elements $\{a_{k_1}, a_{k_2}, \ldots\} \subseteq A$ the inequality holds; $\bigwedge\limits_{m=1}^{\infty} a_{k_m}^{\varepsilon_{k_m}} \neq 0$.*

THEOREM 2B.   *Let $\underline{L} \in PB\sigma A$. The following conditions are equivalent:*

  (i)   *$\underline{L}$ can be weakly imdedded into a Boolean σ-algebra.*
  (ii) *Each formula valid in all Boolean σ-algebras holds in $\underline{L}$.*

(iii) *For every denumerable set $A = \{a_1, a_2, \ldots\} \subseteq L$ and every element $a_{i_0} \in A$ $(a_{i_0} \neq 0)$ there exists a sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ such that*

    1. $\varepsilon_{i_0} = 1$.

    2. For every set of mutually commeasurable elements $\{a_{k_1}, a_{k_2}, \ldots\} \subseteq A$ the inequality holds: $\bigwedge\limits_{m=1}^{\infty} a_{k_m}^{\varepsilon_{k_m}} \neq 0$.

THEOREM 2C. *Let $\underline{L} \in PB\sigma A$. The following conditions are equivalent:*

(i) *$\underline{L}$ can be imbedded into a Boolean σ-algebra.*

(ii) *Each identity $\varphi = \phi$, which holds in all Boolean σ-algebras, is valid in $\underline{L}$.*

(iii) *For every denumerable set $A = \{a_1, a_2, \ldots\} \subseteq L$ and every two distinct elements $a_{i_1}, a_{i_2} \in A$ there exists a sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ such that:*

    1. either $\varepsilon_{i_1} = 1$, $\varepsilon_{i_2} = 0$ or $\varepsilon_{i_1} = 0$, $\varepsilon_{i_2} = 1$

    2. for every set of mutually commeasurable elements $\{a_{k_1}, a_{k_2}, \ldots\} \subseteq A$ the inequality holds: $\bigwedge\limits_{m=1}^{\infty} a_{k_m}^{\varepsilon_{k_m}} \neq 0$.

References

[1] S. Kochen and E. P. Specker, *Logical structure aristing in quantum theory*, [in:] **The theory of models**, ed. by J. W. Addison, L. Henkin and A. Tarski, North-Holland, Amsterdam, 1965.

[2] J. Czelakowski, *Some remarks on transitive partial Boolean algebras*, **Bulletin of the Section of Logic of Inst. of Phil. and Soc. Pol. Acad. Sci.**, vol. 2, No. 3.