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A NOTE ON GENERALIZED KRIPKE-MODELS

Suppose there is a Kripke-semantics which is adequate for the propositional logic $l = (\underline{expr}(l), \underline{taut}(l))$. Then we shall show that l has an adequate Kripke-semantics, the frames of which form an elementary class. Therefore the methods of [3] and [4] can be applied in order to build up a model theory for these logics.

To begin with we have to specify what is meant by an adequate Kripke-semantics. ω denotes the set of all natural numbers. Let $\{p_i : i \in \omega\}$ be a countable set of unary relational symbols. For every similarity type σ let $L(\sigma)$ be the first order language of type σ. For every language L let $F(L)$ be the set of all formulas of L which contain at most the variable x_0 free. A Kripke-semantics is a pair $S = (\underline{K}, \underline{functor})$, where $\underline{K}$ is a class of algebras of an arbitrary but fixed type σ and $\underline{functor}$ is a nonempty subset of $F(L(\sigma \cup \{p_i : i \in \omega\}))$. For every natural number n let $\underline{n - functor} = \underline{functor} \cap L(\sigma \cup \{p_i : i < n\}) - \bigcup\{\underline{m - functor} : m < n\}$. For each $H \in \underline{n - functor}$ let f_H denote a new n-ary propositional connective. The set $\underline{expr}(S)$ of all expressions of the propositional logic defined by S is build up from the propositional variables $p_i (i \in \omega)$ and the propositional connectives f_H $(H \in \underline{functor})$ as usual. A function

$$\underline{meta} : \underline{expr}(S) \to F(L(\sigma \cup \{p_i : i \in \omega\}))$$

is defined by $\underline{meta} - p_i = p_i(x_0)$ for every $i \in \omega$ and $\underline{meta} - f_H \alpha_0 \dots \alpha_{n-1} = H(p_i/\underline{meta} - \alpha_i)$. Here $H(p_i/\underline{meta} - \alpha_i)$ is the formula obtained from H by replacing each subformula of the form $p_i(t)$ by $\underline{meta} - \alpha_i(t)$ where t is some term of $L(\sigma)$.

Define for every $\underline{A} \in \underline{K}$, $a \in |\underline{A}|$, $\alpha \in \underline{expr}(S)$ and for every interpretation $\Vdash$ of the relational symbols $p_i (i \in \omega)$ as subsets of the underlying set of $\underline{A}$ (i.e. $\Vdash (p_i) \subseteq |\underline{A}|$) $a \Vdash \alpha$ if and only if $(\underline{A}, \Vdash, a) \models \underline{meta} - \alpha(a)$.

α is said to be an S-tautology ($\alpha \in \underline{taut}(S)$) if $a \Vdash \alpha$ is true for every $a \in |\underline{A}|$ where $\underline{A} \in \underline{K}$ and for every interpretation (see [1] and [2] for motivation).

Now let $\underline{expr}$ be an arbitrary free algebra with a countable set $\{p_i : i \in \omega\}$ of generators and let $\underline{taut} \subseteq |expr|$. The semantics $S = (\underline{K}, \underline{functor})$ is said to be adequate for the logic $(\underline{expr}, \underline{taut})$ if $(\underline{expr}(S), f_H, \underline{taut}(S))_{H \in \underline{functor}}$ and $(\underline{expr}, \underline{taut})$ are isomorphic.

THEOREM. *If there is a Kripke-semantics which is adequate for the logic $l = (\underline{expr}(l), \underline{taut}(l))$ then there is a Kripke-semantics $S_1 = (\underline{K}_1, \underline{functor})$ and a Kripke-semantics $S_2 = (\underline{K}_2, \underline{functor})$ which are adequate for l and such that $\underline{K}_1$ contains exactly one model and $\underline{K}_2 = \underline{Mod}(\underline{Th}(\underline{K}_1))$.*

PROOF. The proof is a slight modification of the proof of the main theorem of [1]. In [1] it is shown that the assumption of the theorem is equivalent to the existence of a propositional matrix $(M, \varphi, M^*)_{\varphi \in \tau}$ which is adequate for the logic l and such that M^* contains exactly one element ($M^* = \{1\}$) and $M \neq \{1\}$. Let $<$ be an arbitrary linear ordering of M such that 1 is maximal with respect to $<$. We define $\underline{A}(M, \varphi, <, 1)_{\varphi \in \tau}$ such that the interpretation of each $\varphi \in \tau$ in $\underline{A}$ coincides with its interpretation in the propositional matrix $(M, \varphi, M^*)_{\varphi \in \tau}$. For every $\underline{B} \in \underline{Mod}(\underline{Th}(\underline{A}))$, $s \subseteq |\underline{B}|$ $g(s)$ denotes the least element of $|\underline{B}| - s$ if there is such. Otherwise set $g(s) = 1$. Let $\overline{s} = \{a \in |\underline{B}| : a < g(s) \text{ or } a = 1\}$. Obviously we have $g(\overline{s}) = g(s)$. Subsequently $x_{i+1} = g(p_i)$ is an abbreviation of

$$\neg p_i(x_{i+1} \wedge \forall x_{n+1}(x_{n+1} < x_{i+1} \to p_i(x_{i+1})) \vee \neg \exists x_{n+2}(\neg p_i(x_{n+2}) \wedge$$
$$\wedge \forall x_{n+1}(x_{n+1} < x_{n+2} \to p_i(x_{n+2}))) \wedge x_{i+1} = 1.$$

It is clear that for every interpretation $\Vdash a = g(\Vdash (p_i))$ if and only if $(\underline{B}, \Vdash, a) \models a = g(p_i)$. For every n-ary $\varphi \in \tau$ let

$$H_\varphi = \forall x_1 \dots \forall x_n (\bigwedge_{i<n} x_{i+1} = g(p_i) \to x_0 < \varphi(x_1 \dots x_n) \vee x_0 = 1).$$

Now we are ready to define $\underline{functor} = \{H_\varphi : \varphi \in \tau\}$, $S_1 = (\{\underline{A}\}, \underline{functor})$, $S_2 = (\underline{Mod}(\underline{Th}(\underline{A})), \underline{functor})$. For every $\varphi \in \tau$ let $f_{H_\varphi} = \varphi$. If $\underline{B} \in \underline{Mod}(\underline{Th}(\underline{A})), \alpha \in |\underline{expr}(l)|$ for every $i \in \omega$ then we define $s_\alpha(\Vdash) = \{a \in |\underline{B}| : a \Vdash \alpha\}$.

Suppose that $\alpha \notin \underline{taut}(l)$. Then there is a valuation $v : \{p_i : i \in \omega\} \to M$ such that $(\underline{A}, \underline{value}(\alpha, v)) \models \underline{value}(\alpha, v) < 1$. Define $\Vdash (p_i) = \{a \in M :$

$(\underline{A}, v(p_i), a) \models a < v(p_i)\}$ for every $i \in \omega$. It is easy to show by induction on the form of β that $\underline{value}(\beta, v) = g(s_\beta(\Vdash))$ for every $\beta \in |\underline{expr}(l)|$. But this yields $g(s_\alpha(\Vdash)) = \underline{value}(\alpha, v) < 1$ and hence $g(s_\alpha(\Vdash)) \nVdash \alpha$. Thus we have proved that $\underline{taut}(S_1) \subseteq \underline{taut}(l)$ (1).

Assume $\alpha \notin \underline{taut}(S_2)$, i.e. there is a $\underline{B} \in \underline{Mod}(\underline{Th}(\underline{A}))$ and an interpretation $\Vdash$ such that $s_\alpha(\Vdash) \neq |\underline{B}|$. If α is a propositional variable then clearly $\alpha \notin \underline{taut}(l)$ since $M \neq \{1\}$. If $\alpha \notin \{p_i : i \in \omega\}$ then we have $s_\alpha(\Vdash) = \overline{s_\alpha(\Vdash)}$ by the form of the formulae H_φ. Therefore $1 \in s_\alpha(\Vdash)$ and because of $s_\alpha(\Vdash) \neq |\underline{B}|$ there is a least element of $|\underline{B}| - s_\alpha(\Vdash)$. Hence $g(s_\alpha(\Vdash)) < 1$. Let $\underline{B}_\tau$ be the algebra obtained from $\underline{B}$ by cancelling $<$ and the constant 1. Since $\underline{B}$ and $\underline{A}$ are elementary equivalent, $(\underline{B}_\tau, \{1\})$ is a propositional matrix which is adequate for the logic l. A valuation $v : \{p_i : i \in \omega\} \to |\underline{B}|$ is defined by $v(p_i) = g(\Vdash (p_i))$. Again it is easy to show that $\underline{value}(\beta, v) = g(s_\beta(\Vdash))$ for every $\beta \in |\underline{expr}(l)|$. So we obtain $\underline{value}(\alpha, v) = g(s_\alpha(\Vdash)) < 1$. This gives us $\alpha \notin \underline{taut}(l)$ completing our proof that $\underline{taut}(l) \subseteq \underline{taut}(S_2)$ (2).
Now (1), (2) and the obvious inclusion $\underline{taut}(S_2) \subseteq \underline{taut}(S_1)$ give the result desired.

References

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