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DUAL SPACES FOR TOPOLOGICAL BOOLEAN ALGEBRAS

A topological boolean algebra ($t\ b\ a$) $\underline{A} = \langle A; \vee, \wedge, -, 0, 1, I\rangle$ is a boolean algebra $\langle A; \vee, \wedge, -, 0, 1\rangle$ with a unary interior operator I (i.e. $IIa = Ia \leqslant a$, $I1 = 1$, $I(a \wedge b) = Ia \wedge Ib$). We assume that the reader is familiar with the discussion of $t\ b\ a$'s found in [1].

Given $a\ t\ b\ a\ \underline{A}$ let X be the set of prime filters of $\underline{A}$ and τ the topology on X induced by the basis sets $S(a) = \{P \in X | a \in P\}$ for $a \in A$. Clearly $\langle X; \tau\rangle$ is the dual space for $\underline{A}$ as a boolean algebra. Since $I(a \wedge b) = Ia \wedge Ib$, $\{S(Ia) | a \in A\}$ forms a basis for another topology, ι, on X. Obviously ι is coarser than τ, but more than that, the ι-interior of any τ-clopen set is again τ-clopen. We will call $\langle X; \tau, \iota\rangle$ the dual space of $\underline{A}$.

Given a set X and two topologies τ, ι on X we will call $\langle X; \tau, \iota\rangle$ *a t b* space if $\langle X; \tau\rangle$ is a boolean space, ι is coarser than τ, and the ι-interior of any τ-clopen set is τ-clopen. If $\langle X; \tau, \iota\rangle$ is *a t b* space then let A be the boolean algebra of τ-clopen subsets of X. Define I on A to be the ι-interior of any set in A; clearly $\langle A; \cup, \cap, ', \emptyset, X, I\rangle$ is *a t b a*.

For $i = 1, 2$, let $\underline{A}_i$ be *a t b a* with dual space $\langle X_i : \tau_i, \iota_i\rangle$ and let $h : A_1 \to \underline{A}_2$ be a homomorphism. As in the boolean case we define $\varphi : X_2 \to X_1$ by $\varphi(P) = h^{-1}(P)$ for any $P \in X_2$; φ is a continuous map from $\langle X_2; \tau_2\rangle$ to $\langle X_1; \tau_1\rangle$. Also φ satisfies the codnition:

(∗) the inverse image of the ι_1-interior of a τ_1-clopen set equals the ι_2-interior of the inverse image of that τ_1-clopen set.

In particular, φ is a continuous map from $\langle X_2; \iota_2\rangle$ to $\langle X_1; \iota_1\rangle$.

Let $\langle X_i; \tau_i, \iota_i\rangle$ be $t\ b$ spaces and let φ be a satisfies (∗) (such maps will be called compatible). Let $\underline{A}_i$ be the $t\ b\ a$ induced by $\langle X_i; \tau_i, \iota_i\rangle$ and define $h : A_1 \to A_2$ by $h(a) = \varphi^{-1}(a)$ for $a \in A_1$. Then h is a $(t\ b\ a)$ homomorphism.

Thus let TBA be the category of all $t\ b\ a$'s with homomorphisms as morphisms and TBS the category of $t\ b$ spaces with compatible maps as morphisms. A straightforward computation now shows that TBA and TBS are dually naturally equivalent.

A $t\ b\ a$ $\underline{A}$ is self-dual if $I - Ia = -Ia$ (i.e. the closed elements of $\underline{A}$ are also open). Self-dual algebras are equivalent to Halmos' monadic algebras (see [2]). A $t\ b$ space $\langle X; \tau, \iota \rangle$ is dual to a self-dual algebra if and only if the τ-interior of a τ-clopen set is τ-clopen.

In the case that $\underline{A}$ is self-dual an alternate description of ι is possible. Let $\underline{A}$ be a $t\ b\ a$ and let $P, Q \in X$ (the set of prime filters of $\underline{A}$). Set $P \sim Q$ if and only if $\{a \in P | Ia = a\} = \{a \in Q | Ia = a\}$; $\sim$ is an equivalence relation on X. Let $Ker(P)$ be the filter generated by $\{a \in P | Ia = a\}$ and let $[Q]$ be the $\sim$ class containing Q; thus $P \in [Q]$ if and only if $Ker(P) = Ker(Q)$. For $Y \subseteq X$ define $I^*(Y) = \bigcup\{[Q] | [Q] \subseteq Y\}$. It is easily seen that I^* is a self-dual interior operator.

Theorem. *$S(Ia) = I^*(S(a))$ for every $a \in A$ if and only if $\underline{A}$ is self-dual.*

Proof. Clearly it is necessary that $\underline{A}$ be self-dual. For sufficiency, suppose $\underline{A}$ is self-dual. Let $Q \in S(Ia)$; hence $Ia \in Q$. If $R \sim Q$ then $Ia \in R$ so $a \in R$; therefore $[Q] \subseteq S(a)$ so that $Q \in I^*(S(a))$. Conversely let $Q \in I^*(S(a))$; we must show that $Ia \in Q$. Since $\underline{A}$ is self-dual, $Ker(Q)$ is a maximal I-filter. Hence $Ker(Q) = \bigcap\{R | R \in [Q]\}$. But $[Q] \subseteq S(a)$ so that $a \in R$ for all $R \in [Q]$. Thus $a \in Ker(Q)$ and since $Ker(Q)$ is an I-filter, $Ia \in Ker(Q)$. Hence $Ia \in Q$ are desired.

This means that the map $S : a \to S(a)$ is an isomorphism from $\underline{A}$ onto $\langle \{S(a)\}; \cup, \cap, ', \emptyset, X, I^* \rangle$. Thus the duality between TBA and TBS tell us that ι is the topology whose open sets are precisely those sets which are τ-open and I^*-open. That ι need not equal I^* is easily seen by taking $\underline{A}$ infinite and $Ia = a$.

References

[1] H. Rasiowa and R. Sikorski, *The Mathematics of Metamathematics*, **Monografie Matematyczne**, tom 41, Polska Akademia Nauk, Warsaw, 1963.

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