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DEGREE OF MAXIMALITY VERSUS DEGREE OF COMPLETENESS

Define a sentential calculus to be a pair (L, C), where L is a sentential language (algebra of propositional formulas of 0 order, cf. [1]), and C is a structural consequence defined on L, i.e. $eC(X) \subseteq C(eX)$, for every $X \subseteq L$ and for every endomorphism (substitution) e of L.

A set of formulas X closed under substitutions will be called invariant. The cardinality of the set

$$\{C(X) : X \subseteq L, X \text{ is invariant}\}$$

is sometimes called the degree of completeness of (L, C).

Given any consequence C' defined on L, write $C \leqslant C'$, when $C(X) \subseteq C'(X)$, for every $X \subseteq L$. I propose to call the cardinality of the set

$$\{C' : C \leqslant C', C' \text{ is structural }\}$$

the degree of maximality of (L, C).

In order to compare the notion of the degree of completeness with that of the degree maximality consider the three valued Łukasiewicz sentential calculus based on the appropriate set of axioms and modus ponens as the only rule of inference. Denote by Ł the language of this calculus, and by C_3 its consequence operation. We shall assume that Ł consists of all formulas which can be built by means of the connectives: $\rightarrow$ (implication), $\neg$ (negation), $\vee$ (disjunction), and $\wedge$ (conjunction) and propositional variables $p_1, p_2, \ldots, p_i \ldots$ It is known that:

Theorem 1. *The degree of completeness of* $(Ł, C_3)$ *is* 3.

It may be proved that the following holds true:

Theorem 2. *The degree of maximality of* (L, C_3) *is* 4.

The proof of this theorem is rather involved (cf. [2]), and it cannot be presented here. Thus I shall restrict myself to a brief comment.

It is obvious that the set of structural consequence C such that $C_3 \leqslant C$ contains:

1) the consequence Ł defined as follows: $Ł(X) = Ł$, for every X,
2) the consequence C_2 of the two-valued classical sentential logic,
3) the consequence C_3 itself.

The only consequence which belongs to this set and have not been listed above is the consequence $C_3^{\bullet}$ which results by strengthening C_3 be means of the rule defined by the schema:

$$(IN) \qquad \frac{(\alpha \vee \neg\alpha) \rightarrow (\alpha \wedge \neg\alpha)}{\alpha \wedge \neg\alpha}$$

One may prove (cf. [2]) that C_3 is the consequence defined by the logical matrix

$$(Ł, C_3(\emptyset)).$$

References

[1] H. Rasiowa, R. Sikorski, **The metamathematic of mathematics**, PWN.

[2] R. Wójcicki, *The logics stronger than Łukasiewicz's three valued sentential calculus. The notion of degree of maximality versus the notion of degree of completeness*, **Studia Logica**, to appear.

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