Bulletin of the **S**ection of **L**ogic
Volume 3/2 (1974), pp. 11–13

reedition 2012 [original edition, pp. 11–14]

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ON EQUIVALENTAL FRAGMENTS OF SOME INTERMEDIATE LOGICS

This is a summary of a result reported in November 1973 at the seminar of the Department of Logic of Jagiellonian University held by Professor S. J. Surma in Cracow. The full text with detailed proofs will appear in the forthcoming number of Reports on Mathematical Logic.

The symbol F denotes the set of formulas built up in the usual way by means of an infinity of propositional variables: $p, q, \ldots$ and the connectives: $\leftrightarrow, \rightarrow, \wedge, \vee, \neg$ (equivalence, implication, conjunction, disjunction, negation). The symbol Cq denotes the consequence operation in F determined by the theorems of the intuitionistic propositional logic (see [2]) and the detachment rule. By $Sb(X)$ we mean the set of all substitution instances of the formulas of $X \subseteq F$. Let τ_n $(n = 1, 2, \ldots, \omega)$ be the formulas such that:

$\tau_1 = p_1, \tau_{n+1} = (\tau_n \vee (p_n \rightarrow p_{n+1}))$,
$\tau_\omega = (p \rightarrow q) \vee (q \rightarrow p)$.

Each of the well-known intermediate logics $LC_n = Cq(Sb(\{\tau_n\}))$ (see [1] and [5]) determines the consequence operation Cq_n in F such that for every $X \subseteq F$, $Cq_n(X) = Cq(X \cup LC_n)$ $(n = 1, 2, \ldots, \omega)$. Let the symbol Fe denote the set of equivalential formulas of F (i.e. formulas having no occurrence of a connective distinct from $\leftrightarrow$). Putting $Ce(X) = Fe \cap Cq(X)$ and $Ce_n(X) = Fe \cap Cq_n(X)$ for every $X \subseteq Fe$ one defines the consequence operations Ce and Ce_n $(n = 1, 2, \ldots, \omega)$ in the set of equivalential formulas Fe. Analogously, for every $X \subseteq Fe$, $Sbe(X)$ is defined as $Fe \cap Sb(X)$. In order to simplify the notations we shall write down formulas ignoring the equivalence sign $\leftrightarrow$ and abbreviating a formula $(\ldots((\alpha_1\alpha_2)\ldots)\alpha_n)$ by $(\alpha_1 \ldots \alpha_n)$. For example the formula $(((\alpha \leftrightarrow \beta) \leftrightarrow \beta) \leftrightarrow \alpha)$ is abbreviated by $(\alpha\beta\beta\alpha)$ and the formula $((\alpha \leftrightarrow ((\beta \leftrightarrow \gamma) \leftrightarrow \gamma)) \leftrightarrow ((\beta \leftrightarrow \gamma) \leftrightarrow \gamma))$ by $(\alpha(\beta\gamma\gamma)(\beta\gamma\gamma))$. Let the formulas λ, η_n, ϑ_n $(n = 1, 2, \ldots)$ be such that:

$\lambda = ((p(qrr)(qrr))(p(rqq)(rqq))(p(qr)(qr))p)$,
$\eta_1 = p_1, \eta_{n+1} = (p_{n+1}\ \eta_n\ \eta_n\ p_{n+1})$,
$\vartheta_n = (\lambda\ \eta_n)$ (all the propositional variables: $p, q, r, p_1, p_2, \ldots$ are assumed to be distinct). We say that a consequence operation C in Fe satisfies one of the conditions: $(\leftrightarrow)$, (λ), (η_n), (ϑ_n), $n = 1, 2, \ldots$ iff it is such that respectively:

$(\leftrightarrow)$ $C(X \cup \{\alpha\}) = C(X \cup \{\beta\})$ iff $(\alpha\beta) \in C(X)$ for every $X \subseteq Fe$ and $\alpha, \beta \in Fe$,
(λ) $Sbe(\{\lambda\}) \subseteq C(\emptyset)$,
(η_n) $Sbe(\{\eta_n\}) \subseteq C(\emptyset)$,
(ϑ_n) $Sbe(\{\vartheta_n\}) \subseteq C(\emptyset)$.

By a general theorem stated in [4] it follows that:

THEOREM 1. *Ce is the smallest consequence operation in Fe satisfying $(\leftrightarrow)$.*

The consequence operations Ce_n, $n = 1, 2, \ldots, \omega$ can be characterized as follows:

THEOREM 2. *(i) Ce_ω is the smallest consequence operation in Fe satisfying $(\leftrightarrow)$ and (λ); (i) Ce_n, $n = 1, 2, \ldots$ is the smallest consequence operation in Fe satisfying $(\leftrightarrow)$ and (ϑ_n).*

Obviously, if a consequence operation in Fe satisfies $(\leftrightarrow)$ then it satisfies (ϑ_n) iff it satisfies both (λ) and (η_n). Thus, Ce_n, $n = 1, 2, \ldots$ can be characterized equivalently as the smallest consequence operation in Fe satisfying $(\leftrightarrow)$, (λ) and (η_n). Let us note thta $Cq(Sb(\{\lambda\})) = LC_\omega$ and $Cq(Sb(\{\vartheta_n\})) = LC_n$, $n = 1, 2, \ldots$. This fact makes it possible to axiomatize each logic LC_n $(n = 1, 2, \ldots, \omega)$ by means of an equivalential formula. The same is possible for the intermediate logics $LP_n = Cq(Sb(\{\pi\}))$, $n = 1, 2, \ldots$ where $\pi_1 = p_1$ and $\pi_{n+1} = (((p_{n+1} \to \pi_n) \to p_{n+1}) \to p_{n+1})$. The logics LP_n play an important role in the theory of slices developed by Hosoi [3] (LP_n is the smallest logic belonging to the n-th slice). Observing that $(\pi_n\eta_n) \in Cq(\emptyset)$ we get that $LP_n = Cq(Sb(\{\eta_n\}))$, $n = 1, 2, \ldots$.

References

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