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Ewa Graczyńska and Andrzej Wroński

ON THE CONSTRUCTION OF MATRICES STRONGLY ADEQUATE FOR PRE-FINITE LOGICS

This is a summary of the result reported in April 1974 at the XX Conference for the History of Logic in Cracow. The full text with detailed proofs will appear in the forthcoming number of Studia Logica.

In [6] it is stated that no denumerable matrix is strongly adequate for the intuitionistic logic (INT). It is easy to prove that every finite intermediate logic has a finite strongly adequate matrix. Following Maksimowa [5] by pre-finite intermediate logic we mean any intermediate logic for which no finite matrix is adequate matrix. In this paper it is shown that every pre-finite logic has a denumerable strongly adequate matrix.

Let the symbols: $\alpha, \beta_3, \beta_4, \gamma$ denote the formulas: $(x \to y) \vee (y \to x)$, $y \vee (y \to (x \vee \neg x))$, $z \vee (z \to (y \vee (y \to (x \vee \neg x))))$ and $\neg x \vee \neg\neg x$ respectively. In the sequel we will use the terminology and notation taken from [6]. By symbols: LC, LJ, LH we denotes the logics: $Cn(Sb(\alpha))$ (see Dummett [3]), $Cn(Sb(\beta_3))$ (see Jankov [4]), $Cn(Sb(\beta_4, \gamma))$ (see Hosoi-Ono [2]) respectively. $\mathcal{H}$ denotes the two-element Boolean algebra. If $n = 1, 2, \ldots, \omega$ then we will write $n\mathcal{H}$ and $\mathcal{H}^n$ instead of $\underbrace{\mathcal{H} \oplus \ldots \oplus \mathcal{H}}_{n\ times}$ and $\underbrace{\mathcal{H} \times \ldots \times \mathcal{H}}_{n\ times}$ respectively. For the sake of simplicity we use the symbols $\mathcal{A}\oplus$ and $\oplus\mathcal{A}$ instead of $\mathcal{A} \oplus \mathcal{H}$ and $\mathcal{H} \oplus \mathcal{A}$.

Theorem 1.

(i) $LC = E(\omega\mathcal{H})$ *(Dummett [3]);*

(ii) $LJ = E(\mathcal{H}^\omega\oplus)$ *(Jankov [4]);*

(iii) $LH = E(\oplus\mathcal{H}^\omega\oplus)$ *(Hosoi-Ono [2]).*

Theorem 2. *LC, LJ and LH are the only pre-finite intermediate logics (Maksimowa [5]).*

Let $\mathcal{HQ}$ denotes the pseudo-Boolean algebra determined by the chain Q of rational numbers from the closed interval $[0, 1]$ and $\mathcal{F}$ denotes the free ω-generated Boolean algebra.

Theorem 3.

(i) *$\mathcal{HQ}$ is strongly adequate fro LC (i.e. $Cn_{LC} = C_{\mathcal{HQ}}$);*
(ii) *$\mathcal{F}\oplus$ is strongly adequate for LJ (i.e. $Cn_{LJ} = C_{\mathcal{F}\oplus}$);*
(iii) *$\oplus\mathcal{F}\oplus$ is strongly adequate for LH (i.e. $Cn_{LH} = C_{\oplus\mathcal{F}\oplus}$).*

To prove Theorem 3 we will use the criterion of the strong adequacy stated in [6]:

Criterion. *If Δ is an intermediate logic and $\mathcal{A} \in K$ then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is strongly adequate for Δ;*
(ii) *$\mathcal{A} \in K(\Delta)$ and every algebra from $K_0(\Delta)$ is embeddable into $\mathcal{A}$.*

Lemma.

(i) *Every denumerable chain with the first and the last element is embeddable into Q with preservation of these elements (see [1]);*
(ii) *every denumerable Boolean algebra is embeddable into $\mathcal{F}$ (see [1]);*
(iii) *if $\mathcal{A}$ and $\mathcal{B}$ are any pseudo-Boolean algebras and $\mathcal{A}$ is embeddable into $\mathcal{B}$ then $\mathcal{A}\oplus$ is embeddable into $\mathcal{B}\oplus$ and $\oplus\mathcal{A}$ is embeddable into $\oplus\mathcal{B}$.*

The equation $Cn_{LC} = C_{\mathcal{HQ}}$ is a consequence of Lemma(i) and the following facts:

(1) every algebra from $K_0(LC)$ forms a denumerbale chain with the first and the last element (with respect to the lattice ordering);
(2) $\mathcal{HQ} \in K(LC)$.
The equation $Cn_{LJ} = C_{\mathcal{F}\oplus}$ can be obtained from Lemma(ii), (iii) and the facts below:
(3) every algebra from $K_0(LJ)$ is of the form $\mathcal{A}\oplus$ where $\mathcal{A}$ is a denumerable Boolean algebra;
(4) $\mathcal{F}\oplus \in K(LJ)$.

The last equation from Theorem 3 i.e. $Cn_{LH} = C_{\oplus\mathcal{F}\oplus}$ one can derive by means of Lemma(ii), (iii) and the following facts:

(5) if $\mathcal{A}\oplus \in K_0(LH)$ then $y \vee (y \to x) \vee \neg x \in E(\mathcal{A})$ and $A - \{0_{\mathcal{A}}\}$ is the greatest prime filter in $\mathcal{A}$;

(6) if $\mathcal{A}\oplus \in K_0(LH)$ then $\mathcal{A}$ is embeddable into $\oplus\mathcal{B}$ where $\mathcal{B}$ is a denumerable Boolean algebra;

(7) $\oplus\mathcal{F}\oplus \in K(LH)$.

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Section of Logic
Polish Academy of Sciences
Wrocław

and

Department of Logic
Jagiellonian University
Cracow