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A METHOD OF AXIOMATIZATION OF ŁUKASIEWICZ LOGICS

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By the language of Łukasiewicz logics we understand the algebra of formulas $\underline{L} = \langle L, \to, \sim \rangle$. The μ-valued Łukasiwicz matrix $\langle \underline{A}_\mu, \{1\} \rangle$, $\mu = 2, 3, \ldots, \aleph_0$, (cf. [1]), will be denoted here by $\underline{M}_\mu$. $Ł_\mu = R(\underline{M}_\mu)$ is the set of tautologies of $\underline{M}_\mu$ and is called the μ-valued Łukasiewicz system.

All the unexplained notation in this text will come from Wójcicki's paper [4]. Let us define two versions of $\aleph_0$-valued Łukasiewicz consequence:

(A) $C_{\aleph_0}(X)$ is the least set of formulas including $X \cup Ł_{\aleph_0}$ and closed under modus ponens.

(B) $\alpha \in C^*_{\aleph_0}(X)$ iff for every $h : \underline{L} \to^{hom} \underline{A}_{\aleph_0}$ we have $h\alpha = 1$ whenever $h(X) \subseteq \{1\}$.

The following theorem is due to Wójcicki (cf. [4]):

Theorem 1. *For every finite $X \subseteq L$, $C^*_{\aleph_0}(X) = C_{\aleph_0}(X)$.*

Every system $Ł_\mu$, $\mu = 2, 3, \ldots, \aleph_0$ is finitely axiomatizable with modus ponens as the only primitive rule of inference. This result is due to Lindenbaum for the finite case and to Wajsberg for $\aleph_0$ (see the theorems quoted in [1]; cf. also [3]). The schemes of axioms for $Ł_{\aleph_0}$ are the following:

*A*1. $\alpha \to (\beta \to \alpha)$
*A*2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
*A*3. $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$
*A*4. $(\sim \alpha \to \sim \beta) \to (\beta \to \alpha)$

It is the purpose of this paper to give an outline of a new short proof of axiomatizability of $Ł_n$ for every $n \in \omega$, making use of the above Wójcicki's theorem and the McNaughton's criterion for $\underline{M}_{\aleph_0}$ (cf. [2]).

The variable ε is supposed to run over all substitutions in $\underline{L}$: $Sb(X)$ denotes the set of all substitutions of all formulas in $X \subseteq L$; $V(X)$ is the set of all propositional variables occurring in the formulas of X. Finally we put

$$Sb_\alpha(\beta) = \{\varepsilon\beta | \varepsilon : V(L) \to V(\alpha)\}.$$

By virtue of McNaughton's criterion, for every $n \geqslant 2$ there is a formula $\alpha_n(p)$ such that for every $x \in A_{\aleph_0}$

$(*)$ $\qquad \alpha_n(x) = 1$ iff $x \in A_n$.

It is not hard to establish the following lemmas:

LEMMA 1. *For every $n \geqslant 2$, $C^*_{\aleph_0}(Sb(\alpha_n)) = Ł_n$.*

LEMMA 2. *For every $n \geqslant 2$, $\beta \in L$, $\beta \in C^*_{\aleph_0}(Sb(\alpha_n))$ iff $\beta \in C^*_{\aleph_0}(Sb_\beta(\alpha_n))$.*

Since $Sb_\beta(\alpha)$ is a finite set for every $\alpha, \beta \in L$, we obtain

THEOREM 2. *For every $n \geqslant 2$, $C_{\aleph_0}(Sb(\alpha_n)) = Ł_n$.*

PROOF:

$\beta \in Ł_n$ iff $\beta \in C^*_{\aleph_0}(Sb(\alpha_n))$ (by Lemma 1)
iff $\beta \in C^*_{\aleph_0}(Sb_\beta(\alpha_n))$ (by Lemma 2)
iff $\beta \in C_{\aleph_0}(Sb_\beta(\alpha_n))$ (by Theorem 1).

COROLLARY. *If α satisfies $(*)$ then α may be added to $A1$ – $A4$ as the only special axiom scheme for $Ł_n$.*

EXAMPLES. Each one of the following formulas is the only "special" axiom scheme for the suitable system $Ł_n$, if added to $A1 - A4$:

$Ł_2$: $p \vee p, \sim (p \ \& \ \sim p), ((\sim p \to p) \to p) \& ((p \to \sim p) \to \sim p)$.

$Ł_3$: $p \vee \sim p \vee ((p \vee \sim p) \to (p \ \& \ \sim p)), ((\sim p \to p) \& (p \to \sim p)) \vee$
$\sim ((\sim p \to p) \& (p \to \sim p))$.

$Ł_4$: $p \vee \sim p \vee ((p \to \sim (\sim p \to p))) \& (\sim p \to (\sim p \to p))) \vee$
$((\sim p \to \sim (p \to \sim p)) \& (p \to (p \to \sim p)))$.

$Ł_5$: $p \vee \sim p \vee ((p \vee \sim p) \to (p \ \& \ \sim p)) \vee ((\sim p \to (\sim p \to (\sim p \to p))) \&$
$(p \to \sim (\sim p \to (\sim p \to p)))) \vee ((p \to (p \to (p \to \sim p))) \&$
$\sim (p \to \sim (p \to (p \to \sim p))))$.

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