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# $S$ – ALGEBRAS FOR $n$-VALUED SENTENTIAL CALCULI OF ŁUKASIEWICZ

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## 1. Preliminary notions (cf. [4], [5])

Let $S = \langle \underline{F}, C_F \rangle$ be a propositional calculus. Let $\underline{U}$ be an algebra similar to the language $\underline{F}$ and let $m$ be some (selected) element of the universe of $\underline{U}$. By $C_{\underline{U}_m}$ we will denote the consequence operation determined by the matrix $\langle \underline{U}, \{m\} \rangle$. The algebra of the form $\underline{U}_m$ will be called a $S$-*algebra* provided that

$$C_{\underline{F}} \leqslant C_{\underline{U}_m}.$$

If $S = \langle \underline{F}, C_{\underline{F}} \rangle$ is the calculus of the class $\underline{S}$ (class of the standard implicative and extensional systems of sentential calculi – see [4]), then the relation $\approx$ given by

$$\alpha \approx \beta \text{ iff } \alpha \to \beta \in C_{\underline{F}}(\emptyset) \text{ and } \beta \to \alpha \in C_{\underline{F}}(\emptyset)$$

is a congruence relation on the set of all formulas $F$, and the quotient algebra $\underline{A}(S) = \underline{F}/\approx$ is partially ordered by the relation $\leqslant$ defined as follows

$$|\alpha| \leqslant |\beta| \text{ iff } \alpha \to \beta \in C_{\underline{F}}(\emptyset).$$

Lemma 1 (cf. [4]). *The algebra $\underline{A}(S)$ of the consistent system $S = \langle \underline{F}, C_F \rangle \in \underline{S}$ is an $S$-algebra free in the class of all $S$-algebras. The element $m$ which is forwarding in the definition of $S$-algebra here is equal to the maximal element of $\underline{F}/\approx$ (with respect to the order $\leqslant$ defined above). The free generators of the algebra $\underline{A}(S)$ are the classes determined by the sentential variables. The set of all generators will be denoted by $Gen\underline{A}(S)$.*

The pair $S_n = \langle \underline{L}, C_n \rangle$, where $\underline{L} = \langle L, \rightarrow, \vee, \wedge, \neg, \rangle$ is the well known algebra of formulas, and $C_n$ is the consequence operation determined by Łukasiewicz's matrix $\underline{M}_n$ will be called *n-valued sentential calculus of Łukasiewicz*. It can be seen (by a simple testing of the conditions defining the class $\underline{S}$) that the following lemma is valid

LEMMA 2. *For every natural* $n \geqslant 2, S_n \in \underline{S}$.

For this reason we will use the above results concerning the class $\underline{S}$ in dealing with Łukasiewicz's calculi.

## 2. $MV_n$algebras

R. S. Grigolia had introduced the notion of the $MV_n$ algebra in [2] in the following manner

DEFINITION (cf. [2]). We say that the algebra

$$\underline{A} = \langle A, +, \cdot, -, 0, 1 \rangle,$$

where $\underline{A} \neq \emptyset$, and $+, \cdot$ there are the binary operations on $A$; is an $MV_n$ algebra provided that the following conditions are satisfied

$G1$. $\underline{A}$ is an $MV$ algebra (see [1])
$G2$. $(n-1)x + x = (n-1)x$ $G2$. $x^{n-1} \cdot x = x^{n-1}$,

where $1^0$. $0 \cdot x = 0$ and $(m+1)x = mx + x$ and $2^0 \cdot x^0 = 1$ and $x^{m+1} = (x^m) \cdot x$; Moreover – if $n > 3$ then we add the following conditions

$G3$. $\{(jx)(\overline{x} + [(j-1)x]^-)\}^{n-1} = 0$
$G3$. $(n-1)\{x^j + (\overline{x} \cdot [x^{j-1}])\} = 1$,

where $1 \leqslant j \leqslant n-1$ and $j$ is ranging over the set of all natural numbers not dividing $n-1$.

Specially important examples of $MV_n$ algebras are (simple) algebras formed on the bases of Łukasiewicz's matrices. If $\underline{M}_n = \langle A_n, \rightarrow, \vee, \wedge, \neg, \{1\} \rangle$ is an $n$-valued matrix of Łukasiewicz, then the algebra

$$\underline{A}_n = \langle A_n, +, \cdot, -, 0, 1 \rangle,$$

where $x + y = \neg x \rightarrow y$, $x \cdot y = \neg(x \rightarrow \neg y)$ and $\overline{x} = \neg x$ is an $MV_n$ algebra.

R. S. Grigolia had obtained the representation theorem for his $MV_n$ algebras by subdirect products. This theorem is the following

THEOREM 1. (cf. [2]) *Every $MV_n$ algebra $\underline{A}$ is isomorphic with a subdirect product of the algebras $\underline{A}_m$ where $m \leqslant n$ and $m-1$ is a divisor of $n-1$.*

Now let us note a lemma that will be used.

LEMMA 3. *For every natural $n \geqslant 2$, $\underline{A}(S_n)$ is an $MV_n$-algebra.*

## 3. Main theorem

If $n \geqslant 2$ is an arbitrary but fixed natural number, then by $ALG(S_n)$ we denote the class of all $S$-algebras for the calculus $S_n$ ($n$-valued sentential calculus of Łukasiewicz) and we denote by $GMV_n$ the whole class of $MV_n$ algebras. We will prove the following

THEOREM 2. *For every natural $n \geqslant 2$ $ALG(S_n) = GMV_n$.*

PROOF. Let $n \geqslant 2$ be the fixed natural number. From the definition of the $MV_n$ algebra follows that the class $GMV_n$ is equationally definable. We then obtain (from the well known theorem of Birkhoff) that the homomorphic images of $MV_n$ algebras are also $MV_n$ algebras. In particular, if $h$ is a homomorphism, then $h[\underline{A}(S_n)]$ is an $MV_n$ algebra. From the fact that $\underline{A}(S_n)$ is the algebra free in the class of all $S_n$-algebras it follows that an arbitrary mapping of the set $Gen\underline{A}(S_n)$ can be extended to a homomorphism of $MV_n$ algebras.

Now suppose that $\underline{A}^*$ is an $S_n$-algebra and that $\underline{A}^*$ is not an $MV_n$ algebra. Let for some $x_1, x_2, \ldots, x_n$ in $A^*$ (where $A^*$ is carrier of $\underline{A}^*$) and some operation $f$ that can be defined in $A^*$, $f(x_1, x_2, \ldots, x_n)$ does not satisfy an equational condition imposed on $MV_n$ algebra. Let us put also

$$h^* : Gen\underline{A}(S_n) \to B \cup \{x_1, x_2, \ldots, x_n\},$$

where $B \cup \{x_1, x_2, \ldots, x_n\} \subseteq A^*$ and $h^*$ is a mapping such that there exist the sentential variables $p_1, p_2, \ldots, p_n$ for which

$$h|p_1| = x_1$$

$$h|p_2| = x_2$$

$$\cdots\cdots\cdots$$

$$h|p_n| = x_n$$

Then one can prove that the mapping $h^*$ cannot be extended to the homomorphism of $MV_n$ algebras. We obtain the contradiction with Lemma 1 concluding the proof.

Additional Remark. At 1973 were independently introduced two notions of the $MV_n$ algebras (Loosely speaking these are $MV$ algebras corresponding to the finite valued sentential calculi of Łukasiewicz). One of them was introduced by R. S. Grigolia (see [2]) and the other by G. Malinowski (see [3]). The ways in which these two notions were defined there are somewhat different (although these notions are similar). The comparison of the above two kinds of algebras will be given elsewhere. Let us emphasis that in the present paper we have considered only $MV_n$ algebras in the sense of R. S. Grigolia.

## References

[1] C. C. Chang, *Algebraic analysis of many valued logics*, **Trans. Amer. Math. Soc.** 88 (1958), pp. 467–490.

[2] R. S. Grigolia, *Algebraiczeskij naliz n-znacznych logiczeskich sistem Łukasiewicza-Tarskiego*, **Proceedings of Tbilisi University** A 6–7 (149–150), Mathematical an Natural Sciences, Tbilisi (1973), pp. 121–132.

[3] G. Malinowski, *$MV_k$ algebras*, this **Bulletin**, vol. 2, no. 3 (1973), pp. 185–191.

[4] H. Rasiowa, **An algebraic approach to non classical logics**, North-Holland Publ. Co. Amsterdam; PWN, Warszawa, 1974.

[5] R. Wójcicki, *Matrix approach in methodology of sentential calculi*, **Studia Logica**, vol. XXXII (1973), pp. 7–39.

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