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A NOTE ON WEEK FUNCTIONAL COMPLETENESS

This researcz was coarried out in the logical seminar held by Professor R. Wójcicki.

A systematic study of problems of expressibility and functional completeness was begun by A. V. Kuznecov [2] and was recently extended by him in [3]. Using the notion of expressibility M. F. Raca stated in [4] a criterion for functional completeness of any arbitrary intermediate logic.

In this abstract we shall introduce a notion of week expressibility and give a criterion for week functional completeness valid in a certain class of intermediate logics. Our notion of week expressibility is a slight modification of the notion of expressibility due to Kuznecov. It is based on different intuitions and is only formally connected with Kuzniecov's notion.

Definition 1. A finite sequence of formulas $A_1, A_2, \ldots, A_n$ is a week representation of a formula A, in an intermediate logic L by means of a system a set of formulas R if and only if $A_n = A$ and for every A_i, $i \in \{1, 2, \ldots, n\}$ at least one of the following conditions holds:

1. A_i is a propositional variable of L,
2. $A_i \in R$,
3. there exists such $j < i$, $k < i$ and there exists a variable p which occurs in A_k such that A_i is a result of substitution of A_j for all occurrences of p in A_k,
4. there exists some $l < i$, such that $A_l \to A_i$ is a thesis in L.

Definition 2. A formula A is weekly expressible in a logic L by means of a system of formulas R if there exists a week representation of A in L by means of R.

Definition 3. A set of formulas R is week functionally complete in a logic L if all formulas of L are expressible in L by means of R.

Let BL be a pseudo-Boolean algebra corresponding to an intermediate logic L. By $0, 1$ we denote the least and the greatest element in BL, by $\neg$ a pseudo-complementation in BL.

In this paper we shall investigate finite intermediate logics introduces by K. Gödel in [1].

Definition 4. By T we denote the class of all functions $f(x_1, x_2, \dots, x_n)$, $f : BL^n \to BL$ which satisfy the condition

$$f(1, 1, \dots, 1) = 1.$$

Definition 5. By S we denote the class of all functions $f(x_1, x_2, \dots, x_n)$, $f : BL^n \to BL$ which satisfy the condition

$$\neg\neg f(x_1, x_2, \dots, x_n) = \neg f(\neg x_1, \neg x_2, \dots, \neg x_n).$$

Let R be a set of formulas of L. By R_{BL} we shall mean the set of all functions definable in BL by R, i.e., $f(x_1, x_2, \dots, x_n) \in R_{BL}$ if and only if there exists a formula $A(p_1, p_2, \dots, p_n) \in R$ such that for every sequence $a_1, a_2, \dots, a_n$ of elements of BL the following equality holds

$$f(a_1, a_2, \dots, a_n) = A(a_1, a_2, \dots, a_n).$$

Theorem. *In order a system of formulas R be week functionally complete in finite Gödel's logic L it is necessary and sufficient that $R_{BL} \subsetneq T$ and $R_{BL} \subsetneq S$.*

References

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