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MATRIX ŁUKASIEWICZ ALGEBRAS

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In 1940 Gr. C. Moisil introduced (see [1]) the notion of n-valued Łukasiewicz algebra. Intuitions connected with those algebras are explained by the representation theorem (comp. [2]) which, roughly speaking, states that one can treat elements of n-valued Łukasiewicz algebra as increasing n-tuples of elements of a Boolean algebra.

Starting with this observation the author tried to generalize the notion of n-valued algebra of Łukasiewicz in such a way that elements of this generalized algebra could be represented by 'increasing' matrices of elements of a suitable Boolean algebra.

We begin by introducing some useful symbols

0_1. $(n \times m) =^{df} \{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$
0_2. $C(L) =^{df} \{x \in L | \exists y \in L : x \cup y = 1 \; x \cap y = 0\}$
0_3. $C(L)^{(n \times m)} =^{df} \{f : (n \times m) \to C(L)|$ For arbitrary i, j
$r < s$ implies $f(r, j) \subset f(s, j)$ and $f(i, r) \subset f(i, s)\}$

Definition 1. Algebra $\langle L, \{\sigma_{ij}\}_{\langle ij \rangle \ (n \times m)} \rangle$ will be called $n \times m$-valued matrix Łukasiewicz algebra iff

I L is a distributive lattice with the smallest and the greatest elements denoted 0 and 1 respectively.

II $\{\sigma_{ij}\}$ is a family of mutually different endomorphisms of lattice L fulfilling the following conditions:

S1 $\sigma_{ij} : L \to C(L)$
S2 $\sigma_{ij}x \subset \sigma_{ij+1}x$
S3 $\sigma_{ij}x \subset \sigma_{ij+1}x$
S4 $\sigma_{ij}(\sigma_{rs}x) = \sigma_{rs}x$
S5 $\sigma_{ij}0 = 0,\ \sigma_{ij}1 = 1$
S6 If $\forall\langle ij\rangle \in (n \times m) : \sigma_{ij}x = \sigma_{ij}y$ then $x = y$

THEOREM 1. *Every $n \times m$-valued Łukasiewicz algebra L may be embedded in the set $C(L)^{(n\times m)}$.*

The connections of $n \times m$-valued Łukasiewicz algebras and Cartesian products of ordinary Łukasiewicz algebras will be characterized by the theorems stated below.

THEOREM 2. *If $L, L^{|}$ are n-valued and m-valued Łukasiewicz algebras respectively then their product $L \times L'$ with the endomorphisms defined by $\sigma_{ij}\langle x, y\rangle = \langle\sigma_i x, \sigma_j y\rangle$ is a $n \times m$-valued Łukasiewicz algebra.*

THEOREM 3. *If $n \times m$-valued Łukasiewicz algebra L is isomorphic with carteisan product of some two Łukasiewicz algebras n-valued and m-valued respectively, then the following condition $(S6')$ is true:*

If $\sigma_{i1}x = \sigma_{i1}y$ and $\sigma_{1j}x = \sigma_{1j}y$ for every $i \in \{1, \ldots, n-1\}$ and every $j \in \{1, \ldots, m-1\}$ then $x = y$.

We define a useful congruence over L. Let $z \in C(L)$. $x \underset{z}{\sim} y$ iff $x \cap z = y \cap z$.

We put $[x]_z = \{y \in L | x \underset{z}{\sim} y\}$.

THEOREM 4. *$n \times m$-valued Łukasiewicz algebra L is isomorphic with the certain product of n-valued and m-valued algebras of Łukasiewicz iff there exists $z \in C(L)$ such that:*

1^0 *For every $x \in L$ and $i \in \{1, \ldots, n-1\}$*
$[\sigma_{i1}x]_z = \ldots = [\sigma_{im-1}x]_z$

and

2^0 *For every $x \in L$ and $j \in \{1, \ldots, m-1\}$*
$[\sigma_{1j}x]_{\overline{z}} = \ldots = [\sigma_{n-1j}x]_{\overline{z}}$.

Now we will classify the elements occurring in the matrix Łukasiewicz algebras.

DEFINITION 2.

A) Element x will be called
  a) horizontally increasing iff for each $j \in \{1, \ldots, m-2\}$ $\sigma_{n-1j}x \subset \sigma_{1j+1}x$
  b) vertically increasing iff for each $i \in \{1, \ldots, n-2\}$ $\sigma_{im-1}x \subset \sigma_{i+11}x$
  c) increasing iff it is both horizontally and vertically increasing

B) Element x will be called
  a) horizontally inversive iff there exists k such that $\sigma_{1k+1}x \subset \sigma_{n-1k}x$
  b) vertically inversive iff there exists k such that $\sigma_{k+11}x \subset \sigma_{km-1}x$
  c) inversive iff it is both horizontally and vertically inversive

C) Element x will be called
  a) horizontally uncomparable iff there exists k such that $\sigma_{1k+1}x$ is uncomparable with $\sigma_{n-1k}x$
  b) vertically uncomparable iff there exists k such that $\sigma_{k+11}x$ is uncomparable with $\sigma_{km-1}x$
  c) uncomparable iff it is both horizontally and vertically uncomparable.

THEOREM 5.

a) Every vertically uncomparable element which is not horizontally uncomparable is horizontally inversive.

b) Every horizontally uncomparable element which is not vertically uncomparable is vertically inversive.

THEOREM 6.

a) The set of vertically increasing elements of algebra L, is closed under operations of L.

b) The set of horizontally increasing elements of algebra L, is closed under operations of L.

c) The set of increasing elements of algebra L, is closed under operations of L.

Theorem 7. *If L is an $n \times m$-valued Łukasiewicz algebra then the set of vertically increasing elements and the set of horizontally increasing elements are embeddable in some $[(n-1).(m-1)+1]$-valued Łukasiewicz algebra.*

In the class of matrix Łukasiewicz algebras, like in ordinary-Łukasiewicz algebras, the centred ones play special role.

Definition 3. In $n \times m$-valued algebra of Łukasiewicz the element c_{ij} will be called the $\langle ij \rangle$-centre of this algebra provided that

$$\sigma_{rs} c_{ij} = \begin{cases} 0 & i > r \quad \text{or} \quad j > s \\ 1 & i \leqslant r \quad \text{and} \quad j \leqslant s \end{cases}$$

Hereafter the symbol c_{ij} is reserved for $\langle ij \rangle$-centre.

Definition 4. $n \times m$-valued Łukasiewicz algebra L will be called centred iff for every pair $\langle ij \rangle \in (n \times m)$ there exists the $\langle ij \rangle$-centre of L.

Lemma. *Every element x from $n \times m$-valued centred Łukasiewicz algebra may be represented as* $\bigcup_{i=1}^{n-1} \bigcup_{j=1}^{m-1} (c_{ij} \cap \sigma_{ij} x)$.

Theorem 8. *$n \times m$-valued Łukasiewicz algebra L is centred iff it is isomorphic with $C(L)^{(n \times m)}$.*

The next group of noteworthy matrix Łukasiewicz algebras are square-matrix Łukasiewicz algebras. In these algebras one can introduce two new operations, one of them is negation.

Definition 5. $n \times n$-valued algebra of Łukasiewicz will be called symmetric iff for every element x the set $V_x^N \neq \emptyset$. ($V_x^N =^{df} \{z \in L | \forall \langle rs \rangle \in (n \times n) : \sigma_{rs} z = \overline{\sigma_{n-s\ n-r} x}\}$).

In such algebras one can define operation N, called negation, putting: $Nx = z$ iff $z \in V_x^N$.

Definition 6. $n \times n$-valued Łukasiewicz algebra will be called involutive iff for every its element x the set $V_x^S \neq \emptyset$. ($V_x^S =^{df} \{z \in L | \forall \langle rs \rangle \in (n \times n) : \sigma_{rs} z = \sigma_{sr} x\}$).

In the involutive algebras one can define the operation S called symmetry putting: $Sx = z$ iff $z \in V_x^S$. Now the connections between the notions introduced lately and the centred algebra will be explained.

THEOREM 9. *Every centred $n \times n$-valued Łukasiewicz algebra is both symmetric and involutive.*

THEOREM 10. *Let $n > 2$. In each $n \times n$-valued symmetric Łukasiewicz algebra there is no element equal to its negation.*

DEFINITION 7. Let L be $n \times n$-valued Łukasiewicz involutive algebra. We put $Z(L) = \{y \in L | y = Sy\}$.

LEMMA.

α) *If $Sx = Nx$ then $x \notin Z(L)$.*
β) *If n is even and $n > 2$ then there is no x such that $Nx = Sx$.*

THEOREM 11. *If L is $n \times n$-valued involutive Łukasiewicz algebra then $Z(L)$ is closed under the operations $\cup, \cap, S, \sigma_{ij}$ and when L is symmetric then $Z(L)$ is closed under N, too.*

The last two theorems characterize the behaviour of the operations N and S over the sets of elements defined by Def. 2.

THEOREM 12. *Operation N leads elements*

a) verticaly increasing in horizontaly increasing and conversely
b) verticaly inversive in horizontaly inversive and conversely
c) verticaly uncomparable in horizontaly uncomparable and conversely.

THEOREM 13. *Operation S leads elements*

a) horizontally increasing in vertically increasing and conversely
b) vertically inversive in horizontally inversive and conversely
c) horizontally uncomparable in vertically uncomparable and conversely.

References

[1] Gr. C. Moisil, *Notes sur les logiques non chrysippiennes*, **Annales Scientifiques**, Jassy 27 (1941), pp. 80–98.

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[4] T. Traczyk, **Wstęp do teorii algebr Boole'a**, PWN 1970, Warszawa.

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