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REPRESENTATION THEOREM FOR DISTRIBUTIVE PSEUDO-BOOLEAN ALGEBRA

This is an abstract of the paper which will be presented at the Conference for Logical Calculi (Wrocław, November 22-24).

1. Distributive pseudo-Boolean algebras

We shall say that an abstract algebra $\underline{A} = (A, \cup, \cap, \Rightarrow, \neg)$ is a pseudo-Boolean algebra provided that

(i) $(A, \cup, \cap)$ is a lattice with the zero element,
(ii) is a binary operation which satisfies the following condition: $x \leqslant a \Rightarrow b$ if and only if $x \cap a \leqslant b$ for any $a, b, x \in A$,
(iii) $\neg$ is an unary operation defined by the formula

$$\neg a = a \Rightarrow \bigwedge$$

1.1. In every pseudo-Boolean algebra $\underline{A} = (A, \cup, \cap, \Rightarrow, \neg)$ $a \leqslant b$ iff there exists $c \in A$ such that $a \leqslant b \cup c$ and $a \cap c \leqslant b$ for every $a, b \in A$.

1.2. Let $\underline{A} = (A, \cup, \cap, \Rightarrow, \neg)$ be a pseudo-Boolean algebra. If an infinite join $\bigcup_{t\in T} a_t$ exists in $\underline{A}$, then for every $a \in A$ the join $\bigcup_{t\in T} a_t \cap a$ also exists in $\underline{A}$ and

$$a \cap \bigcup_{t\in T} a_t = \bigcup_{t\in T} a_t \cap a \tag{1}$$

If an infinite meet $\bigcap_{t\in T} b_t$ exists in a pseudo-Boolean algebra, then for every a $\underline{A}$ the meet $\bigcap_{t\in T}(a \cup b_t)$ also exists in A and

$$a \cup \bigcap_{t\in T} b_t \leqslant \bigcap_{t\in T}(b_t \cup a). \tag{2}$$

The proof of 1.2 can be found in [2].

It is well known that the sign $\leqslant$ in (2) can not be replace by $=$.

We shall call a distributive pseudo-Boolean algebra any pseudo-Boolean algebra such that

$$(D) \qquad a \cup \bigcap_{t\in T} b_t = \bigcap_{t\in T}(b_t \cup a)$$

provided that all infinite meets exists.
For simplify our notation these algebras will be denoted by $DPBA$.

An example for a $DPBA$ can be constructed in the following way. By an quasi-ordered set we mean an ordered pair $\underline{C} = \langle C, \leqslant\rangle$ – where C is a non-empty set, $\leqslant$ is a transitive and reflexive relation on C. Let B C. We call B open if whenever $x \in B$ and $x \leqslant y$ then $y \in B$. We take for $\underline{0}_{\leqslant}$ the collection of all open subsets of $\underline{C}$ and for the ordering relation $\leqslant$ we take set-theoretical inclusion. We note that the algebra $(\underline{0}_{\leqslant}, \cup, \cap)$ – where the operations $\cup$ and $\cap$ are just the ordinary union and intersection, respectively – is a distributive lattice with the zero element θ (θ – the empty set). Now, let $\Rightarrow, \neg$ be two new operations in $\underline{0}_{\leqslant}$ defined by the formulas:

(3) $R \Rightarrow S = \{c \in C : \sqcap c' \in C(c \leqslant c' \wedge c' \in R \to c' \in S),$
(4) $\neg R = \{c \in C : \sqcap c' \in C(c \leqslant c' \wedge c' \notin R),$

for every $R, S \in \underline{0}_{\leqslant}$.

By an easy verification we can prove the following

1.3. The algebra $\underline{0}_{\leqslant} = (\underline{0}_{\leqslant}, \cup, \cap, \Rightarrow, \neg)$ – where $\underline{0}_{\leqslant}$ is the family of all open sets of a quasi-ordered set $\underline{C}$, the operations $\cup$ and $\cap$ are set-theoretical union and intersection, respectively, the operations $\Rightarrow$ and $\neg$ are defined by (3) and (4), respectively, – is a complete $DPBA$.

Theorem 1.3 yields an important example of a $DPBA$. In the sequel, this algebra will be called an order topology. This example is typical in this sense that we have the following representation theorem for $DPBA$:

Theorem 1. *For every DPBA $\underline{A}$ there exist an order topology $\underline{0}_{\leqslant}$ and monomorphism h from $\underline{A}$ to $\underline{0}_{\leqslant}$.*

The proof of Theorem 1 is the same as the proof of analogous theorem for pseudo-Boolean algebra [1].

2. Q-filters in $DPBA$

Let $\underline{A} = (A, \cup, \cap, \Rightarrow, \neg)$ be a $DPBA$ algebra, and let (Q) be a set of infinite joins and meets in $\underline{A}$:

$$(Q)\quad \begin{array}{ll} a_{2n} = \bigcup_{a \in A_{2n}} a & n \in \omega \\ b_{2n+1} = \bigcap_{b \in B_{2n+1}} b & n \in \omega \end{array}$$

A prime filter ∇ is said to be a Q-filter provided that

(f_1) for every $n \in \omega$ if $a_{2n} \in \nabla$ then $A_{2n} \cap \nabla \neq 0$,

(f_2) for every $n \in \omega$ if $B_{2n+1} \subset \nabla$ then $b_{2n+1} \in \nabla$.

The next theorem is analogous to the Rasiowa-Sikorski lemma.

Theorem 2. *Let $\underline{A} = (A, \cup, \cap, \Rightarrow, \neg)$ be a $DPBA$ and let the set (Q) be defined as above. Let x, y be the elements of A such that the relation $x \leqslant y$ does not hold. Then there exists a Q-filter ∇ such that $x \in nabla$ and $y \notin \nabla$.*

Proof. We can define two sequences $\langle \alpha_n, n \in \omega \rangle$ and $\langle \beta_n, n \in \omega \rangle$ of the elements of A such that:

(i) $\alpha_0 = y$ $\beta_0 = x$,

(ii) $\alpha_{n-1} \leqslant \alpha_n$ and $\beta_{n-1} \geqslant \beta_n$ for $n > 0$,

(iii) either $\beta_{2n+1} \leqslant b_{2n+1}$ or there exists $b \in B_{2n+1}$ such that $b \geqslant \alpha_{2n+1}$ for every $n \in \omega$, either there exists $a \in A_{2n}$ such that $\beta_{2n} \leqslant a$ or $a_{2n} \leqslant \alpha_{2n}$ for every $n \in \omega$,

(iv) for every $n \in \omega$ the relation $\beta_n \leqslant \alpha_n$ does not hold.

Let I be the ideal generated by the sequence $\langle \alpha_n : n \in \omega \rangle$ and F be the filter generated by $\langle \beta_n : n \in \omega \rangle$. Then by (iv) I and F are disjoint and

(v) either $b_{2n+1} \in F$ or there exists $b \in B_{2n+1}$ such that $b \in I$, for $n \in \omega$,

(vi) either there exists $a \in A_{2n}$ such that $a \in F$ or $a_{2n} \in I$, for any $n \in \omega$.

It is well known that in a distributive lattice, every filter can be separated from an ideal disjoint from it by a prime filter. Let ∇ be a prime filter containing F such that is disjoint from I. It is obvious that $x \in \nabla$ and $x \notin \nabla$. By (v) and (iv) ∇ is the required Q-filter, which completes the proof of Theorem 2.

We observe that if for every $n \in \omega$ $a_{2n} = \bigcup_{a \in A_m} a$ and $b_{2n+1} = \bigcap_{b \in B_{2n+1}} b$ exist then

$$\bigcap_{a\in A_{2n}} (a \Rightarrow c) \qquad \bigcap_{a\in A_{2n}} ((a \cap d) \Rightarrow c)$$

$$\bigcap_{b\in B_{2n+1}} (c \Rightarrow b) \qquad \bigcap_{b\in B_{2n+1}} (c \Rightarrow (b \cup d))$$

also exist for every $n \in \omega$, $c, d \in A$.

2.1. Let $\underline{A} = (A, \cup, \cap, \Rightarrow, \neg)$ be a $DPBA$ and suppose that for every $n \in \omega$, A_{2n}, $B_{2n+1} \subset A$ and

(i) a_{2n} and b_{2n+1} exist,

(ii) for any $c \in A$, $\{a \Rightarrow c | a \in A_{2n}\} \in \{B_{2k+1} | k \in \omega\}$
$\{c \Rightarrow b | b \in B_{2n+1}\} \in \{B_{2k+1} | k \in \omega\}$

(iii) for any $c, d \in A$, $\{(a \cap d) \Rightarrow c | a \in A_{2n}\} \in \{B_{2k+1} | k \in \omega\}$
$\{c \Rightarrow (b \cup d) | b \in B_{2n+1}\} \in \{B_{2k+1} | k \in \omega\}$

Let ∇ be a Q-filter in $\underline{A}$. For every $a \in A$, let $|a|$ denote an element of A/∇. Then the relation $\leqslant$ defined by the formula

$$|a| \leqslant |b| \text{ iff } a \Rightarrow b \in \nabla$$

is the lattice ordering in A/∇ and

$$|a| \cup |b| = |a \cup b| \qquad |a| \Rightarrow |b| = |a \Rightarrow b|$$

$$|a| \cap |b| = |a \cap b| \qquad \neg|a| = nega|$$

.

If the infinite join $\bigcup_{a\in A_{2n}} |a|$ exists in $\underline{A}$ then $\bigcup_{a\in A_{2n}} a$ exists in A/∇ and

$\bigcup_{a\in A_{2n}} |a| = |\bigcup_{a\in A_{2n}} a|$.

If the infinite meet $\bigcap_{b\in B_{2n+1}} b$ exists in $\underline{A}$ then the infinite meet $\bigcap_{b\in B_{2n+1}} |b|$ exists in A/∇ and

$\bigcap_{b\in B_{2n+1}} |b| = |\bigcap_{b\in B_{2n+1}} b|$.

Moreover, the algebra A/∇ is a $DPBA$.

2.2. Let $\underline{A}$ be a $DPBA$ and suppose that for $n \in \omega$ A_{2n}, $B_{2n+1} \subset A$ such that (1) – (iii) from 2.1 are satisfied. Let ∇ be a Q-filter such that $a \Rightarrow b \notin \nabla$. Then there exists a Q-filter ∇' such that $a \in \nabla'$, $b \notin \nabla'$ and $\nabla \subset \nabla'$.

Let $\underline{A} = (A, \cup, \cap, \Rightarrow, \neg)$ be a $DPBA$ and let ∇ be a Q-filter such that $a \Rightarrow b \notin \nabla$. We take quotient algbera A/∇. By 2.1 this algebra is a $DPBA$.

We note that relation $|a| \leqslant |b|$ does not hold. On account Theorem 2 there exists a Q-filter $\widetilde{\nabla}$ such that $|a| \in \widetilde{\nabla}$ and $|b| \notin \widetilde{\nabla}$. Let us set

$$\nabla' = \{x \in A : |x| \in \widetilde{\nabla}\}.$$

It is obvious that ∇' is a filter. Moreover, ∇' is a Q-filter as $\widetilde{\nabla}$ is a Q-filter. We observe that $a \in \nabla'$ and $b \notin \nabla'$. Now, let $x \in \nabla$, then $|x| = V_{A/\nabla}$ and $|x| \in \widetilde{\nabla}$. It gives that $x \in \nabla'$ which proves that $\nabla \subset \nabla'$, i.e. ∇' is the required Q-filter.

We denote by ζ the set of all Q-filter of a $DPBA$. We take $\nabla \in \zeta$.

2.3. $a \Rightarrow b \in \nabla$ iff for every $\nabla' \in \zeta$ such that $\nabla \subset \nabla'$ if $a \in \nabla'$ then $b \notin \nabla'$. This lemma follows from 2.1.

THEOREM 3. *For every $DPBA$ $\underline{A}$ there exist an order topology $\underline{0}_{\leqslant}$ and a monomorphism h from $\underline{A}$ to $\underline{0}_{\leqslant}$ preserving a_{2n} and b_{2n+1}, for $n \in \omega$.*

We take for $\underline{0}_{\leqslant}$ the class of all Q-filters of a $DPBA$ $\underline{A}$ and for $\leqslant$ the set inclusion. Let h (a) be the set of all Q-filters $\nabla \in \underline{0}_{\leqslant}$ such that $a \in \nabla$. In standard way using Theorem 2 and 2.2 we show that h is the required monomorphism from A to $\underline{0}_{\leqslant} = (\underline{0}_{\leqslant}, \cup, \cap, \Rightarrow, \neg)$ – where the operations $\Rightarrow$ and $\neg$ are defined by (3) and (4), respectively.

References

[1] M. C. Fitting, **Intuitionistic Logic Model Theory and Forcing**, North-Holland, Amsterdam-London 1969.

[2] H. Rasiowa and R. Sikorski, **The Mathematics of the Metamathematics**, PWN, Warsaw 1963.

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