Bulletin of the **S**ection of **L**ogic
Volume 3:3/4 (1974), pp. 24–26

reedition 2012 [original edition, pp. 24–26]

Paweł Bielak

ON FUNCTIONS DEFINIABLE IN IMPLICATIONAL ALGEBRAS

This paper was presented at the seminar held by Professor R. Wójcicki.

In this abstract we shall give a criterion for definiability of functions of arbitrary arity in implicational algebras.

The case of binary functions definiable in finite implicational Gödel's algebras was discussed by M. Tokarz in [2].

In shall deal with some subclass of implicational algebras as defined in [1]. By implicational algebra we shall understand an abstract algebra $\underline{I} = \langle I, V, \to \rangle$ with 0-argument operation V, and two argument operation $\to$. The set I is a chain ordered by $\leqslant$ with the greatest element V. The operation $\to$ is defined as follows:

$$a \to b = \begin{cases} V & \text{if} \quad a \leqslant b \\ b & \text{if} \quad a > b \end{cases}$$

Let $\underline{L} = (L, \to)$ be a sentential language corresponding to I, where L is the set of all formulas built up by means of sentential variables $p_1, p_2, \ldots, p_n, \ldots$ and the connective $\to$. Every homomorphism h, of $\underline{L}$ into $\underline{I}$ will be called a valuation (of formulas) of $\underline{L}$ in $\underline{I}$.

Let $A(p_1, p_2, \ldots, p_n)$ be a formula of $\underline{L}$ built up by means of exactly the variables $p_1, p_2, \ldots, p_n$. We shall denote by $(A(a_1, a_2, \ldots, a_n)$ the value of A under the valuation h such that $kp_1 = a_1$, $hp_2, = a_2, \ldots$, $hp_n = a_n$, i.e.

$$A(a_1, a_2, \ldots, a_n) = h(A(p_1, p_2, \ldots, p_n)).$$

Definition 1. A function $f(x_1, x_2, \ldots, x_n)$, $f : I^n \to I$, is definiable in I if there is a formula $A(p_1, p_2, \ldots, p_n)$ of $\underline{L}$ such that for every sequence

$a_1, a_2, \ldots, a_n$ of elements of I the equality holds

$$f(a_1, a_2, \ldots, a_n) = A(a_1, a_2, \ldots, a_n).$$

LEMMA 1. (see [2]). *If $f(x_1, x_2, \ldots, x_n)$ is definiable in some implicational algebra then there exists such an index i, $1 \leqslant i \leqslant n$, that for every sequence $a_1, a_2, \ldots, a_n$ of elements of the algebra*

$$f(a_1, a_2, \ldots, a_n) \in \{a_i, V\}.$$

DEFINITION 2. We shall say that the function $f(x_1, x_2, \ldots, x_n)$ depends mainly on the variable x_i if i fulfils the conclusion of the Lemma 1.

DEFINITION 3. Let us consider a sequence $\underline{a} = (a_1, a_2, \ldots, a_n)$ of elements of I. Let i be any but fixed number among $1, 2, \ldots, n$ such that $a_i = V$. The sequence $\underline{b} = (b_1, b_2, \ldots, b_n)$ of elements of $I(a \sim_i b)$ if:

1. $b_i \neq V$.
2. If for $a_r, a_s \leqslant a_i$ we have $a_r = a_s$ or $a_r < a_s$ then $b_r = b_s$ or $b_r < b_s$ respectively.
3. For any b_k such that $k \in \{k : a_k > a_i\}$ we have $b_k > b_i$.

LEMMA 2. *The relation of i-similarity of sequences is an equivalence.*

LEMMA 3. *For any but fixed numbers n, i, $1 \leqslant i \leqslant n$ and $S = \{(a_1, a_2, \ldots, a_n) : a_1, a_2, \ldots, a_n,\ I,\ a_i \neq V\}$ the quotient set $S/\sim_i$ is finite.*

THEOREM 1. *A function $f : I^n \to I$ is definable in I if and only if the following conditions are satisfied:*

1. *There exists a variable x_i such that $f(x_1, x_2, \ldots, x_n)$ depends mainly on x_i, $1 \leqslant i \leqslant n$.*
2. *If $f(a_1, a_2, \ldots, a_n) = a_i \neq V$ and f depends mainly on x_i then for every sequence $(b_1, b_2, \ldots, b_n)$ which is i-similar to sequence $(a_1, a_2, \ldots, a_n)$ we have $f(b_1, b_2, \ldots, b_n) = b_i$.*

THEOREM 2. *For any function $f : I^n \to I$ satisfying the conditions 1,2 of Theorem 1 there exists an effective procedure for finding a formula which defines f.*

References

[1] H. Rasiowa, **An algebraic approach to non-classical logic**, Warszawa 1974.

[2] M. Tokarz, *Binary functions definiable in implicational Gödel algebra"*, **Bulletin of the Section of Logic** *PAN, vol. 3/1 (1974), pp. 22–24.*

Department of Logic
Wrocław University