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THE IDENTITY RELATION AND PARTIAL BOOLEAN ALGEBRAS

Let $P_0 = \{x_1, x_2, \ldots\}$ be a countable set of variables and $\underline{P} = \langle P; \vee, \neg\rangle$ be an absolutely free algebra in the variables $x_1, x_2, \ldots$ and connectives $\vee, \neg$. If $\varphi, \psi \in P$, then the expression $\varphi = \psi$ is called an identity.

Let $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$ PBA (PBA denote the class of all partial Boolean algebras, see [1]). According to Kochen and Specker [2] we say:

1^0 "The identity $\varphi = \psi$ weakly holds in $\underline{B}$" iff for every valuation $h : \underline{P} \to \underline{B}$ (see [4]) if $\varphi, \psi \in Dom(h)$ and $h\varphi \overset{|}{\circ} h\psi$, then $h\varphi = h\psi$.

2^0 "The identity $\varphi = \psi$ strongly holds in $\underline{B}$" (or short: "$\varphi = \psi$ holds in $\underline{B}$") iff for every valuation $h : \underline{P} \to \underline{B}$ if $\varphi, \psi \in Dom(h)$, then $h\varphi = h\psi$.

If $\underline{B}\langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$ is established, then we write $\varphi \sim_B \psi$ when $\varphi = \psi$ weakly holds in $\underline{B}$ and $\varphi \approx_B \psi$ when $\varphi = \psi$ holds in $\underline{B}$.

$\sim_B$ and $\approx_B$ are always symmetric and reflexive and $\varphi \approx_B \psi$ implies $\varphi \sim_B \psi$. Notice that if $\underline{B}$ is a Boolean algebra, then $\sim_B$ coincides with $\approx_B$ and $\sim_B$ ($=\approx_B$) is a congruence in $\underline{P}$.

We shall say that $\varphi = \psi$ is a Boolean identity iff $\varphi = \psi$ holds in all Boolean algebras.

The following theorems are direct consequences of the Theorem 2 given in 3.

Theorem A: *Let* $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$ PBA. *The following conditions are equivalent:*

1. $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$ *is embeddable into a Boolean algebra.*
2. *Each Boolean identity strongly holds in* $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$.

3. $\approx_B$ *is the equivalence relation in* $\underline{P}$.

Theorem B: *Let* $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$ *PBA. The following conditions are equivalent:*

1. $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$ *is weakly embeddable into a Boolean algebra.*
2. *Each Boolean identity weakly holds in* $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$.
3. $\approx_B$ *is the equivalence relation in* $\underline{P}$.

In the proof of implication 3. $\rightarrow$ 1. of Theorem B we make use of the following relation $\overset{\sim}{\approx}^{B}$ in $\underline{P}$: $\varphi \overset{\sim}{\approx}^{B} \psi$ iff there exists a finite sequence of identities $\varphi_1 = \psi_1, \varphi_2 = \psi_2, \ldots, \varphi_n = \psi_n$ of the properties:

(a) for every j $(1 \leqslant j \leqslant n)$ $\varphi_j = \psi_j$ strongly holds in $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$
(b) $\varphi_1 = \varphi, \psi_i\ \varphi_{i+1}$ for every i $(1 \leqslant i \leqslant n-1)$, $\psi_n = \psi$.

Under the assumption 3^0 $\overset{\sim}{\approx}^{B}$ is a congruence in $\underline{P}$ and $P/\overset{\sim}{\approx}^{B}$ is a non-trivial Boolean algebra. Notice that $\overset{\sim}{\approx}^{B}$ is contained in $\sim_B$. Next we use the following fact: $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$ is weakly embeddable into a Boolean algebra iff for every tautology (of the Classical Propositional Calculus) of the form $\varphi \leftrightarrow \psi$, the identity $\varphi = \psi$ weakly holds in $\underline{B}$.

The above theorems show that the relations $\approx_B$, $\sim_B$ need not be transitive.

References

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[4] J. Czelakowski, *Partial Boolean σ-algebras*, **Bulletin of the Section of Logic** of Inst. of Phil. and Soc. Pol. Acad. Sci., Vol. 3, No. 1.

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