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CHARACTERIZATION OF MEDVEDEV'S LOGIC BY MEANS OF KUBIŃSKI'S FRAMES

Abstract

In the paper we deal with the frames introduced by T. Kubiński, and show that the intersection of their content coincides with the well known Medvedev's logic of finite problems.

Key words: Kubiński's lattice, Medvedev's logic of finite problems, Kripke semantics, p-morphism.

1. Kubiński's lattice $\mathcal{R}_n$

T. Kubiński in [3] presented an original theory of vague terms. One of his basic assumption is that the denotation of a vague term c is a pair (A, B), where A and B are disjoint subsets of fixed universe U (universe of all objects) and the elements of these sets are treated as undoubted references of c and as undoubted non-references of c, respectively. The family of all such pairs forms a lattice which we call here Kubiński's lattice.

In the Sections 1, 2 and 3 we deal with the finite universes U. The last section is devoted to the infinite case.

For any natural n, put $U_n = \{1, 2, \ldots, n\}$ and $P_n = \mathcal{P}(U_n) \times \mathcal{P}(U_n)$; the set P_n is partially ordered by the relation:

$$(A, B) \leq (C, D) \;\Leftrightarrow\; A \subseteq C \;\&\; D \subseteq B.$$

It is easy to show that $(P_n, \leq)$ is the Boolean lattice with zero – $(\emptyset, U_n)$, unit – $(U_n, \emptyset)$, supremum:

$$(A, B) \vee (C, D) = (A \cup C, B \cap D),$$

and infimum:

$$(A, B) \wedge (C, D) = (A \cap C, B \cup D).$$

The lattice $(P_n, \leq)$ is atomic: the atoms have the forms:

$$(\emptyset, U_n \setminus \{i\}) \quad or \quad (\{i\}, U_n), \qquad where\ i \in U_n,$$

so we have:

Lemma 1. $(P_n, \leq) \cong (\mathcal{P}(U_{2n}), \subseteq)$.

The proper subset of P_n which consists of the pairs (A, B) whose coordinates are disjoint:

$$R_n = \{(A, B) \in P_n : A \cap B = \emptyset\},$$

with $\leq$ restricted, forms the *Kubiński's lattice* $\mathcal{R}_n = (R_n, \leq)$. Supremum, infimum, zero and unit are exactly like in the earlier case. Moreover, the lattice $\mathcal{R}_n$ is still distributive, but not complementary: for $A, B \subseteq U_n$ disjoint and $A \cup B \neq U_n$, there is no complementation of (A, B).[1] Figure 1 presents the Hasse diagram of lattices $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ (eg. $12, \emptyset$ denotes the element $(\{1, 2\}, \emptyset)$).

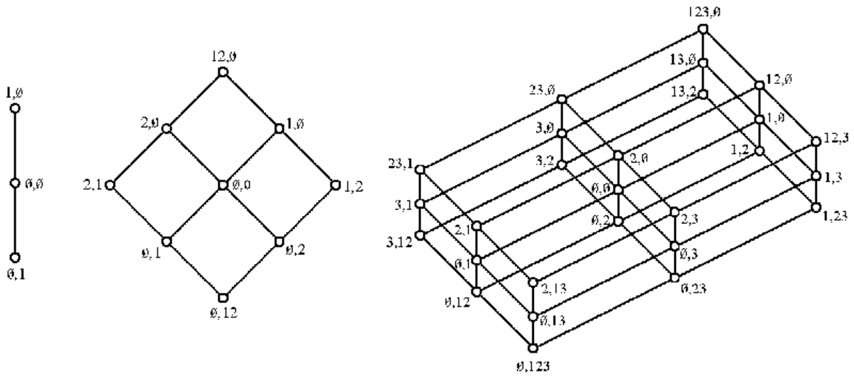

Figure 1: Kubiński's lattices $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$

[1]In terms of gluing, we can describe the lattice $\mathcal{R}_n$ as the so called *Wroński's sum* based on a skeleton which is n-dimensional Boolean cube, and *scare* representation which consists of n-dimensional Boolean cubes, too (for details see [1], [2]).

2. Kubiński's logic KL

By Kubiński's logic we mean here intermediate logic determined by frames $\mathcal{K}_n = (K_n, \leq)$ obtained from lattices $\mathcal{R}_n$ by deleting the unit:

$$K_n = R_n \smallsetminus \{(U_n, \emptyset)\},$$

and restricting the relation $\leq$. More precisely, we consider a propositional language $\langle Fm, \wedge, \vee, \rightarrow, \neg \rangle$, generated by the set of propositional variables $At \subseteq Fm$. In the standard way, we define the intuitionistic forcing relation $\Vdash_e$ determined by valuation $e\colon At \rightarrow \mathcal{P}(K_n)$ (the map such that $e(p)$ is upward closed, for any $p \in At$), and the set of tautologies of $\mathcal{K}_n$:

$$\boldsymbol{L}(\mathcal{K}_n) = \{\alpha \in Fm : x \Vdash_e \alpha, \ \textit{for every valuation } e \textit{ and } x \in K_n\}.$$

Finally, we obtain Kubiński's logic KL:

$$KL = \bigcap \{\boldsymbol{L}(\mathcal{K}_n) : n > 0\}.$$

We should add that Kubiński neither in [3] nor elsewhere considers his lattices as Kripke frames, so our term "Kubiński's logic" is rather a convention.

3. Kubiński's and Medvedev's logic coincide

We will employ a semantical characterizaton of Medvedev's logic of finite problems ML (see [4]). Let us consider the frames $\Sigma_n = (\sigma_n, \subseteq)$, where $\sigma_n = \mathcal{P}(U_n) \smallsetminus \{U_n\}$. Then ML is the set of these formulas which are tautologies in every frame Σ_n:

$$ML = \bigcap \{\boldsymbol{L}(\Sigma_n) : n > 0\}.$$

ML is the only known structurally complete intermediate logic with the disjunction property (see [5]). It remains an open question whether ML is decidable (comp. [6]). Our main result is the following:

THEOREM 1. $KL = ML$.

The proof of this claim is purely semantical: we deal with frames, K_n and Σ_m, constructing appropriate p-morphisms between them. Let us recall this notion and some facts.

Let $\mathcal{V} = (V, \leq_V, \Vdash_V)$, $\mathcal{W} = (W, \leq_W, \Vdash_W)$ be the Kripke models. The surjecion $M\colon V \to W$ is said to be *p-morphism* iff the following conditions hold:

$$x \leq_V y \;\Rightarrow\; M(x) \leq_W M(y) \tag{1}$$

$$M(x) \leq_W z \;\Rightarrow\; \exists y \in V\ (z = M(y)\ \&\ x \leq_V y) \tag{2}$$

$$x \Vdash_V p \;\Leftrightarrow\; M(x) \Vdash_W p \tag{3}$$

for each $x, y \in V$, $z \in W$ and $p \in At$. We use a simple induction on the length of formula α in order to prove:

Lemma 2. *If $M\colon V \to W$ is p-morphism of Kripke models $\mathcal{V} = (V, \leq_V, \Vdash_V)$ and $\mathcal{W} = (W, \leq_W, \Vdash_W)$, then for every $x \in V$ and formula α holds:*

$$x \Vdash_V \alpha \;\Leftrightarrow\; M(x) \Vdash_W \alpha.$$

We can also express the same claim in a more convenient form:

Lemma 3. *Let $\mathcal{V} = (V, \leq_V)$, $\mathcal{W} = (W, \leq_W)$ be Kripke frames and $M\colon V \to W$ be surjection fulfilling conditions* (1) *and* (2) *of the definition of p-morphism. Then*

$$\boldsymbol{L}(\mathcal{V}) \subseteq \boldsymbol{L}(\mathcal{W}).$$

To prove Theorem 1 it is sufficient to show the following

Lemma 4. *For each natural $n > 0$:*

1. $\boldsymbol{L}(K_n) \subseteq \boldsymbol{L}(\Sigma_n)$.
2. $\boldsymbol{L}(\Sigma_{2n}) \subseteq \boldsymbol{L}(K_n)$.

Proof. Due to Lemma 3 we need to construct suitable surjections fulfilling conditions (1) and (2).

For (i) we easily check that the map $M\colon K_n \to \sigma_n$ defined as follows:

$$M(A, B) = A,$$

is the one we need.

To prove (ii) let $S_n = P_n \smallsetminus \{(U_n, \emptyset)\}$; hence by Lemma 1 we have

$$(S_n, \leq) \cong (\sigma_{2n}, \subseteq),$$

so we shall be done if we show that the map $M\colon S_n \to K_n$ given by:

$$M(A, B) = (A \smallsetminus B, B \smallsetminus A),$$

satisfies the appropriate conditions.

First of all, it is easy to see that it is indeed a surjection onto K_n. It is also clear that condition (1) holds.

To prove (2) let us fix $(A, B) \in S_n$, $(C, D) \in K_n$ (so $C \cap D = \emptyset$) and assume that $M(A, B) \leq (C, D)$. Hence we achieve:

$$A \smallsetminus B \subseteq C, \quad D \subseteq B \smallsetminus A,$$

so:

$$C \cap D = \emptyset, \quad A \cap B' \cap C' = \emptyset, \quad D \cap B' = \emptyset, \quad D \cap A = \emptyset, \tag{4}$$

(where $'$ stands for the set-theoretical complement operator w.r.t. U_n). Our goal is to point out $(E, F) \in S_n$ such that:

$$A \subseteq E, \quad F \subseteq B, \quad E \smallsetminus F = C, \quad F \smallsetminus E = D. \tag{5}$$

Let us consider all the components of A, B, C, D. Then by (4) we have:

$X_1 = A \cap B \cap C \cap D = \emptyset$, $X_9 = A \cap B' \cap C' \cap D = \emptyset$,
$X_2 = A \cap B \cap C \cap D'$, $X_{10} = A' \cap B \cap C' \cap D$,
$X_3 = A \cap B \cap C' \cap D = \emptyset$, $X_{11} = A' \cap B' \cap C \cap D = \emptyset$,
$X_4 = A \cap B' \cap C \cap D = \emptyset$, $X_{12} = A \cap B' \cap C' \cap D' = \emptyset$,
$X_5 = A' \cap B \cap C \cap D = \emptyset$, $X_{13} = A' \cap B \cap C' \cap D'$,
$X_6 = A \cap B \cap C' \cap D'$, $X_{14} = A' \cap B' \cap C \cap D'$,
$X_7 = A \cap B' \cap C \cap D'$, $X_{15} = A' \cap B' \cap C' \cap D = \emptyset$,
$X_8 = A' \cap B \cap C \cap D'$, $X_{16} = A' \cap B' \cap C' \cap D'$.

Hence the sets A, B, C, D can be presented as follows:

$A = X_2 \cup X_6 \cup X_7$, $C = X_2 \cup X_7 \cup X_8 \cup X_{14}$,
$B = X_2 \cup X_6 \cup X_8 \cup X_{10} \cup X_{13}$, $D = X_{10}$.

Putting:

$E = X_2 \cup X_6 \cup X_7 \cup X_8 \cup X_{14}$, $F = X_6 \cup X_{10}$,

it is easy to see that all conditions (5) are true, which completes the proof. ■

The basic idea of the proof of Lemma 4 (ii) is the appropriate tagging of σ_{2n}. Figure 2 presents this situation for $n = 2$.

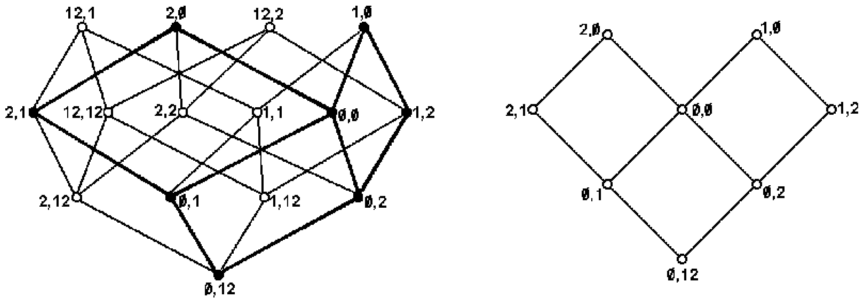

Figure 2: p-morphism $M\colon \sigma_4 \to K_2$

4. Infinite Kubiński's frames

In [6] Skvortsov considers the *logic of infinite problems*:

$$ML' = \bigcap\{\boldsymbol{L}(\Sigma(U)) : U \textit{ is arbitrary set}\},$$

where $\Sigma(U) = (\sigma(U), \subseteq)$ and $\sigma(U) = \mathcal{P}(U) \setminus \{U\}$. Obviously $ML' \subseteq ML$; moreover, it has been proved that

$$ML' = \boldsymbol{L}(\Sigma(\omega)), \tag{6}$$

where ω stands for the set of all naturals (see [6], Theorem 2.2).

The *infinite Kubiński's frame* is $\mathcal{K}(U) = (K(U), \leq)$, where $K(U)$ denotes the family of all pairs of disjoint subsets of U except the pair $(U, \emptyset)$. The relation $\leq$ on $K(U)$ is defined as in Section 1. Let us consider the logic

$$KL' = \bigcap\{\boldsymbol{L}(\mathcal{K}(U)) : U \textit{ is arbitrary set}\}.$$

The technique of the proof of Lemma 4 can be applied to the following

LEMMA 5. *For an arbitrary infinite set U:*

1. $\boldsymbol{L}(\mathcal{K}(U)) \subseteq \boldsymbol{L}(\Sigma(U))$.
2. $\boldsymbol{L}(\Sigma(U)) \subseteq \boldsymbol{L}(\mathcal{K}(U))$.

PROOF. The part (i) is immediate. For (ii) let us observe that since U is infinite we have:

$$(\mathcal{P}(U) \times \mathcal{P}(U), \leq) \cong (\mathcal{P}(U), \subseteq).$$

Hence the map $M\colon \mathcal{P}(U) \times \mathcal{P}(U) \smallsetminus \{(U, \emptyset)\} \to K(U)$ given by

$$M(A, B) = (A \smallsetminus B, B \smallsetminus A),$$

satisfies the appropriate conditions of Lemma 3. ■

Finally, by Lemma 5 and (6) we obtain

THEOREM 2. $ML' = KL' = \boldsymbol{L}(\mathcal{K}(\omega))$.

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