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PSEUDO-BCI-LOGIC

Abstract

A non-commutative version of the BCI-logic, pseudo-BCI-logic, is introduced. Although it is not algebraizable, it is extended to logic which is so. The main result of the paper says that a pseudo-BCI-algebra is an algebraic counterpart of this extended logic (Theorem 3.2).

Keywords and phrases: pseudo-BCI-logic, pseudo-BCI-algebra, algebraizability of logic

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1. Introduction

The BCI-logic, mentioned by A. N. Prior in [11], is attributed to C. A. Meredith and dated in 1956. Its significance is due to a certain correspondence between combinators and implicational formulas (see [2] and [10]). The BCI-logic is the propositional logic with the axioms:

(B) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$,
(C) $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$,
(I) $\alpha \to \alpha$

and the only inference rule:

(MP): $\frac{\alpha, \alpha \to \beta}{\beta}$.

In 1966 K. Iséki introduced the concept of BCI-algebras as an algebraic counterpart of the BCI-logic (see [5]). Unfortunately, BCI-algebras fails to be the models of the BCI-logic. W. J. Blok and D. Pigozzi proved that the BCI-logic is not algebraizable (see Theorem 5.9 of [1]). A BCI-algebra is an algebraic counterpart of the BCI-logic extended on one additional inference rule (see [7]):

(Imp): $\frac{\alpha,\beta}{\alpha\to\beta}$.

In this paper we present a non-commutative version of the BCI-logic, pseudo-BCI-logic $ps\mathcal{BCI}$. Although it is not algebraizable, we easily extend it to logic $ps\mathcal{BCI}'$ which is so. Moreover, we show that pseudo-BCI-algebras are the models of logic $ps\mathcal{BCI}'$, which is the main result of the paper. We do this similarly as it is done in [8] for pseudo-BCK-logic. The reader should also be familiar with [1].

2. Pseudo-BCI-algebras

A *pseudo-BCI-algebra* is a structure $\mathcal{X} = (X, \leq, \to, \rightsquigarrow, 1)$, where $\leq$ is a binary relation on a set X, $\to$ and $\rightsquigarrow$ are binary operations on X and 1 is an element of X such that for all $x, y, z \in X$, we have

(a1) $x \to y \leq (y \to z) \rightsquigarrow (x \to z), x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$,
(a2) $x \leq (x \to y) \rightsquigarrow y, x \leq (x \rightsquigarrow y) \to y$,
(a3) $x \leq x$,
(a4) if $x \leq y$ and $y \leq x$, then $x = y$,
(a5) $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$.

It is obvious that any pseudo-BCI-algebra $(X, \leq, \to, \rightsquigarrow, 1)$ can be regarded as a universal algebra $(X, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$. Note that every pseudo-BCI-algebra satisfying $x \to y = x \rightsquigarrow y$ for all $x, y \in X$ is a BCI-algebra. Notice also that every pseudo-BCI-algebra satisfying $x \leq 1$ for all $x \in X$ is a pseudo-BCK-algebra.

Now we list some basic properties of pseudo-BCI-algebras from [3], [6] and [9]. Let $\mathcal{X}$ be a pseudo-BCI-algebra. The following holds for all $x, y, z \in X$:

(b1) if $1 \leq x$, then $x = 1$,
(b2) if $x \leq y$, then $y \to z \leq x \to z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,
(b3) if $x \leq y$ and $y \leq z$, then $x \leq z$,
(b4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,

(b5) $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$,
(b6) $x \to y \leq (z \to x) \to (z \to y), x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,
(b7) if $x \leq y$, then $z \to x \leq z \to y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,
(b8) $1 \to x = 1 \rightsquigarrow x = x$,
(b9) $((x \to y) \rightsquigarrow y) \to y = x \to y, ((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$,
(b10) $x \to y \leq (y \to x) \rightsquigarrow 1$,
(b11) $x \rightsquigarrow y \leq (y \rightsquigarrow x) \to 1$,
(b12) $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \rightsquigarrow 1)$,
(b13) $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \to 1)$,
(b14) $x \to 1 = x \rightsquigarrow 1$.

Remark. If $\mathcal{X} = (X, \leq, \to, \rightsquigarrow, 1)$ is a pseudo-BCI-algebra, then, by (a3), (a4), (b3) and (b1), $(X, \leq)$ is a poset with 1 as a maximal element.

The class of pseudo-BCI-algebras forms a quasivariety:

Lemma 2.1. *An algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is a pseudo-BCI-algebra if and only if it satisfies the following identities and quasi-identity:*

(i) $(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = 1$,
(ii) $(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1$,
(iii) $1 \to x = x$,
(iv) $1 \rightsquigarrow x = x$,
(v) $x \to y = 1 \;\;\&\;\; y \to x = 1 \;\Rightarrow\; x = y$.

Proof: Every pseudo-BCI-algebra obviously satisfies (i)–(v). Conversely, assume that an algebra $\mathcal{X}$ satisfies (i)–(v). Putting $x = 1$, $y = 1$ and $z = x$ in (i) and (ii) and using (iii) and (iv), we have

$$1 = (1 \rightsquigarrow 1) \to [(1 \rightsquigarrow x) \to (1 \rightsquigarrow x)] = x \to x$$

and

$$1 = (1 \to 1) \rightsquigarrow [(1 \to x) \rightsquigarrow (1 \to x)] = x \rightsquigarrow x.$$

So, (a3) is satisfied. Now, putting $x = 1$, $y = x$ and $z = y$ in (i) and (ii) we get, by (iii) and (iv),

$$1 = (1 \to x) \rightsquigarrow [(x \to y) \rightsquigarrow (1 \to y)] = x \rightsquigarrow [(x \to y) \rightsquigarrow y]$$

and

$$1 = (1 \rightsquigarrow x) \to [(x \rightsquigarrow y) \to (1 \rightsquigarrow y)] = x \to [(x \rightsquigarrow y) \to y].$$

Hence, (a2) is also satisfied. Further, if $x \to y = 1$, then, by (iv), $x \rightsquigarrow y = x \rightsquigarrow (1 \rightsquigarrow y) = x \rightsquigarrow [(x \to y) \rightsquigarrow y] = 1$, and analogously, if $x \rightsquigarrow y = 1$, then, by (iii), $x \to y = x \to (1 \to y) = x \to [(x \rightsquigarrow y) \to y] = 1$. Thus, $x \to y = 1$ iff $x \rightsquigarrow y = 1$. It is therefore easily seen that the relation $\leq$ is defined by

$$x \leq y \text{ iff } x \to y = 1 \text{ iff } x \rightsquigarrow y = 1$$

making the structure $(X, \leq, \to, \rightsquigarrow, 1)$ into a pseudo-BCI-algebra. □

Remark. Since pseudo-BCI-algebras include BCI-algebras, which are not closed under homomorphic images (see [12]), it follows that the quasivariety of pseudo-BCI-algebras is not a variety.

3. Pseudo-BCI-logic

In this section we present pseudo-BCI-logic, a non-commutative version of BCI-logic. Following Hájek's definition of his basic logic (see [4]), definition of pseudo-BCI-logic is as follows:

The formulas of *pseudo-BCI-logic* ($ps\mathcal{BCI}$, for short) are built from propositional variables and the primitive connectives $\to$ and $\rightsquigarrow$. The following formulas are the axioms of $ps\mathcal{BCI}$ (where α, β and γ are arbitrary formulas):

(B1) $(\alpha \to \beta) \to ((\beta \to \gamma) \rightsquigarrow (\alpha \to \gamma))$,
(B2) $(\alpha \rightsquigarrow \beta) \to ((\beta \rightsquigarrow \gamma) \to (\alpha \rightsquigarrow \gamma))$,
(C1) $(\alpha \to (\beta \rightsquigarrow \gamma)) \to (\beta \rightsquigarrow (\alpha \to \gamma))$,
(C2) $(\alpha \rightsquigarrow (\beta \to \gamma)) \to (\beta \to (\alpha \rightsquigarrow \gamma))$,
(I) $\alpha \to \alpha$.

The inference rules are:

(MP): $\frac{\alpha, \alpha \to \beta}{\beta}$,

(Imp1): $\frac{\alpha \to \beta}{\alpha \rightsquigarrow \beta}$,

(Imp2): $\frac{\alpha \rightsquigarrow \beta}{\alpha \to \beta}$.

Remark. Using advanced methods and techniques of [1] it can be proved that the logic $ps\mathcal{BCI}$ is not algebraizable (particularly see Theorem 5.9 of [1]).

In order to be algebraizable, we have to extend pseudo-BCI-logic on the inference rule:

(Imp): $\frac{\alpha,\beta}{\alpha\to\beta}$.

The extended logic, *pseudo-BCI'-logic* ($ps\mathcal{BCI}'$, for short) has the axioms: (B1), (B2), (C1), (C2) and (I), and the inference rules: (MP), (Imp1), (Imp2) and (Imp).

Next theorem shows the algebraizability of the logic $ps\mathcal{BCI}'$ (in the sense of [1]).

THEOREM 3.1. *The logic* $ps\mathcal{BCI}'$ *is algebraizable with the set of equivalence formulas* $\triangle = \{x \to y, y \to x\}$ *and defining equation* $x = x \to x$.

PROOF: Following the notation of [1], we write $\alpha\triangle\beta$ as an abbreviation of $\{\alpha \to \beta, \beta \to \alpha\}$ for any formulas α, β. In order to show that $ps\mathcal{BCI}'$ is algebraizable, by Theorem 4.7 of [1], we have to prove the following properties, for all formulas $\alpha, \beta, \gamma, \alpha_1, \beta_1$ (for the convenience we write $\vdash$ instead of $\vdash_{ps\mathcal{BCI}'}$):

(i) $\vdash \alpha\triangle\alpha$,
(ii) $\alpha\triangle\beta \vdash \beta\triangle\alpha$,
(iii) $\alpha\triangle\beta, \beta\triangle\gamma \vdash \alpha\triangle\gamma$
(iv) $\alpha\triangle\beta, \alpha_1\triangle\beta_1 \vdash (\alpha \to \alpha_1)\triangle(\beta \to \beta_1), (\alpha \rightsquigarrow \alpha_1)\triangle(\beta \rightsquigarrow \beta_1)$,
(v) $\alpha \dashv\vdash \alpha\triangle(\alpha \to \alpha)$.

(i): It is immediate consequence of (I).

(ii): It is trivial, because $\alpha\triangle\beta = \beta\triangle\alpha$.

(iii): By (B1), $\alpha\triangle\beta \vdash (\beta \to \gamma) \rightsquigarrow (\alpha \to \gamma)$. Hence, $\alpha\triangle\beta, \beta\triangle\gamma \vdash (\alpha \to \gamma)$. Now, replacing α and γ we get $\alpha\triangle\beta, \beta\triangle\gamma \vdash (\gamma \to \alpha)$. Thus (iii) holds.

(iv): From (B1) and (Imp2) it follows $\alpha\triangle\beta \vdash (\alpha \to \alpha_1) \to (\beta \to \alpha_1)$ and $\alpha\triangle\beta \vdash (\beta \to \alpha_1) \to (\alpha \to \alpha_1)$. So,

$$\alpha\triangle\beta \vdash (\alpha \to \alpha_1)\triangle(\beta \to \alpha_1). \tag{1}$$

By (Imp1), $\alpha\triangle\beta \vdash (\alpha \rightsquigarrow \beta)$ and $\alpha\triangle\beta \vdash (\beta \rightsquigarrow \alpha)$. Hence, by (B2), $\alpha\triangle\beta \vdash (\alpha \rightsquigarrow \alpha_1) \to (\beta \rightsquigarrow \alpha_1)$ and $\alpha\triangle\beta \vdash (\beta \rightsquigarrow \alpha_1) \to (\alpha \rightsquigarrow \alpha_1)$. Thus,

$$\alpha\triangle\beta \vdash (\alpha \rightsquigarrow \alpha_1)\triangle(\beta \rightsquigarrow \alpha_1). \tag{2}$$

Further, by (B1), $\vdash (\beta \to \alpha_1) \to ((\alpha_1 \to \beta_1) \rightsquigarrow (\beta \to \beta_1))$ and $\vdash (\beta \to \beta_1) \to ((\beta_1 \to \alpha_1) \rightsquigarrow (\beta \to \alpha_1))$. Hence, by (C1), $\vdash (\alpha_1 \to \beta_1) \rightsquigarrow ((\beta \to \alpha_1) \to (\beta \to \beta_1))$ and $\vdash (\beta_1 \to \alpha_1) \rightsquigarrow ((\beta \to \beta_1) \to (\beta \to \alpha_1))$. Thus,

$$\alpha_1 \triangle \beta_1 \vdash (\beta \to \alpha_1) \triangle (\beta \to \beta_1). \tag{3}$$

Similarly, by (B1) and (Imp1), $\vdash (\beta \rightsquigarrow \alpha_1) \rightsquigarrow ((\alpha_1 \rightsquigarrow \beta_1) \to (\beta \rightsquigarrow \beta_1))$ and $\vdash (\beta \rightsquigarrow \beta_1) \rightsquigarrow ((\beta_1 \rightsquigarrow \alpha_1) \to (\beta \rightsquigarrow \alpha_1))$. Hence, by (C2), $\vdash (\alpha_1 \rightsquigarrow \beta_1) \to ((\beta \rightsquigarrow \alpha_1) \rightsquigarrow (\beta \rightsquigarrow \beta_1))$ and $\vdash (\beta_1 \rightsquigarrow \alpha_1) \to ((\beta \rightsquigarrow \beta_1) \rightsquigarrow (\beta \rightsquigarrow \alpha_1))$. Thus, $\alpha_1 \triangle \beta_1 \vdash (\beta \rightsquigarrow \alpha_1) \rightsquigarrow (\beta \rightsquigarrow \beta_1)$ and $\alpha_1 \triangle \beta_1 \vdash (\beta \rightsquigarrow \beta_1) \rightsquigarrow (\beta \rightsquigarrow \alpha_1)$ and so, by (Imp2), $\alpha_1 \triangle \beta_1 \vdash (\beta \rightsquigarrow \alpha_1) \to (\beta \rightsquigarrow \beta_1)$ and $\alpha_1 \triangle \beta_1 \vdash (\beta \rightsquigarrow \beta_1) \to (\beta \rightsquigarrow \alpha_1)$. Therefore,

$$\alpha_1 \triangle \beta_1 \vdash (\beta \rightsquigarrow \alpha_1) \triangle (\beta \rightsquigarrow \beta_1). \tag{4}$$

Finally, by (iii), (1) and (3), we obtain

$$\alpha \triangle \beta, \alpha_1 \triangle \beta_1 \vdash (\alpha \to \alpha_1) \triangle (\beta \to \beta_1)$$

and similarly, by (iii), (2) and (4) we get

$$\alpha \triangle \beta, \alpha_1 \triangle \beta_1 \vdash (\alpha \rightsquigarrow \alpha_1) \triangle (\beta \rightsquigarrow \beta_1)$$

which end the proof of (iv).

(v): To prove (v) we must verify:

(a) $\alpha \vdash \alpha \to (\alpha \to \alpha)$,
(b) $\alpha \vdash (\alpha \to \alpha) \to \alpha$,
(c) $\alpha \to (\alpha \to \alpha), (\alpha \to \alpha) \to \alpha \vdash \alpha$.

(a): We have it by (I) and (Imp).

(b): By (i) and (Imp1), $\vdash (\alpha \to \alpha) \rightsquigarrow (\alpha \to \alpha)$, so by (C2), $\vdash \alpha \to ((\alpha \to \alpha) \rightsquigarrow \alpha)$. Hence, $\alpha \vdash (\alpha \to \alpha) \rightsquigarrow \alpha$ and, by (Imp2), $\alpha \vdash (\alpha \to \alpha) \to \alpha$. Thus (b) holds.

(c): By (i) and (Imp1) we have $\vdash ((\alpha \to \alpha) \to \alpha) \rightsquigarrow ((\alpha \to \alpha) \to \alpha)$, which implies, by (C2), $\vdash (\alpha \to \alpha) \to ((\alpha \to \alpha) \to \alpha) \rightsquigarrow \alpha$. Since, by (i), $\vdash \alpha \to \alpha$, it follows, by (MP), $\vdash ((\alpha \to \alpha) \to \alpha) \rightsquigarrow \alpha$ and, by (Imp2), $\vdash ((\alpha \to \alpha) \to \alpha) \to \alpha$. Thus, (c) also holds.

Therefore, the logic $ps\mathcal{BCI}'$ is algebraizable. □

The *equivalent quasivariety semantics* (see [1]) for the logic $ps\mathcal{BCI}'$ is a quasivariety $\mathcal{I}$ of algebras $(X, \to, \rightsquigarrow)$ of type $(2, 2)$ satisfying certain identities and quasi-identities, which are derived from the axioms and inference rules of $ps\mathcal{BCI}'$ using $\triangle = \{x \to y, y \to x\}$ and $x = x \to x$, such that

(i) for every set of formulas Σ and every formula α,

$$\Sigma \vdash_{ps\mathcal{BCI}'} \alpha \ \text{ iff } \ \{\beta = \beta \to \beta : \beta \in \Sigma\} \models_{\mathcal{I}} \alpha = \alpha \to \alpha,$$

(ii) for every formulas α, β,
$\alpha = \beta \dashv\models_{\mathcal{I}} \{\alpha \to \beta = (\alpha \to \beta) \to (\alpha \to \beta), \beta \to \alpha = (\beta \to \alpha) \to (\beta \to \alpha)\}$.

Notice that $\models_{\mathcal{I}} \alpha \to \beta = (\alpha \to \beta) \to (\alpha \to \beta)$ iff $\vdash_{ps\mathcal{BCI}'} \alpha \to \beta$, and similarly, $\models_{\mathcal{I}} \beta \to \alpha = (\beta \to \alpha) \to (\beta \to \alpha)$ iff $\vdash_{ps\mathcal{BCI}'} \beta \to \alpha$. Thus,

$$\models_{\mathcal{I}} \alpha = \beta \ \text{ iff } \ (\vdash_{ps\mathcal{BCI}'} \alpha \to \beta \text{ and } \vdash_{ps\mathcal{BCI}'} \beta \to \alpha) \ \text{ iff } \ \vdash_{ps\mathcal{BCI}'} \alpha \triangle \beta.$$

Next theorem is the main result of the paper and it says that the class of pseudo-BCI-algebras forms an algebraic semantics for the logic $ps\mathcal{BCI}'$.

THEOREM 3.2. *The quasivariety of pseudo-BCI-algebras is definitionally equivalent to the equivalent quasivariety semantics for the logic* $ps\mathcal{BCI}'$.

PROOF: First, note that by (I) and (Imp) we have $\vdash (\alpha \to \alpha) \to (\beta \to \beta)$ and $\vdash (\beta \to \beta) \to (\alpha \to \alpha)$. Thus, $\vdash (\alpha \to \alpha)\triangle(\beta \to \beta)$. Analogously, using additionally (Imp1), we obtain that $\vdash (\alpha \to \alpha)\triangle(\alpha \rightsquigarrow \alpha)$ and $\vdash (\alpha \rightsquigarrow \alpha)\triangle(\beta \rightsquigarrow \beta)$. Hence, the equivalent algebraic semantics $\mathcal{I}$ satisfies the identities $x \to x = y \to y = y \rightsquigarrow y$. Thus, every algebra $(X, \to, \rightsquigarrow)$ in $\mathcal{I}$ possesses a constant 1 such that $1 = x \to x = x \rightsquigarrow x$ for all $x \in X$. Let $\mathcal{I}^*$ be the class consisting of algebras $(X, \to, \rightsquigarrow, 1)$ such that $(X, \to, \rightsquigarrow)$ belongs to $\mathcal{I}$. Using Theorem 2.17 of [1], we get that the quasivariety $\mathcal{I}^*$ is axiomatized as follows:

(1) $(x \to y) \to ((y \to z) \rightsquigarrow (x \to z)) = 1$,
(2) $(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1$,
(3) $(x \to (y \rightsquigarrow z)) \to (y \rightsquigarrow (x \to z)) = 1$,
(4) $(y \rightsquigarrow (x \to z)) \to (x \to (y \rightsquigarrow z)) = 1$,
(5) $x \to x = 1$,
(6) $x = 1 \ \& \ x \to y = 1 \ \Rightarrow \ y = 1$,
(7) $x \to y = 1 \ \Rightarrow \ x \rightsquigarrow y = 1$,
(8) $x \rightsquigarrow y = 1 \ \Rightarrow \ x \to y = 1$,
(9) $x = 1 \ \& \ y = 1 \ \Rightarrow \ x \to y = 1$,
(10) $x \to y = 1 \ \& \ y \to x = 1 \ \Rightarrow \ x = y$.

It is obvious that every pseudo-BCI-algebra satisfies (1)–(10). Hence, the quasivariety of pseudo-BCI-algebras is included in $\mathcal{I}^*$.

Conversely, let $(X, \to, \rightsquigarrow, 1)$ be an algebra belonging to $\mathcal{I}^*$. From Lemma 2.1 it suffices to show the following equations

$$1 \to x = x \quad \text{and} \quad 1 \rightsquigarrow x = x.$$

From (3), (4) and (10) we get the following identity

$$x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z).$$

Hence, by (5) and (7), $1 \to ((1 \to x) \rightsquigarrow x) = (1 \to x) \rightsquigarrow (1 \to x) = 1$ and $1 \rightsquigarrow ((1 \rightsquigarrow x) \to x) = (1 \rightsquigarrow x) \to (1 \rightsquigarrow x) = 1$. Thus, by (6) and (7), $(1 \to x) \rightsquigarrow x = 1$ and $(1 \rightsquigarrow x) \to x = 1$, and so, by (8), $(1 \to x) \to x = 1$ and $(1 \rightsquigarrow x) \to x = 1$. On the other hand, by (5), (7) and (8), $x \to (1 \to x) = x \rightsquigarrow (1 \to x) = 1 \to (x \rightsquigarrow x) = 1 \to 1 = 1$ and $x \to (1 \rightsquigarrow x) = 1 \rightsquigarrow (x \to x) = 1 \rightsquigarrow 1 = 1$. Thus, by (10), $1 \to x = x$ and $1 \rightsquigarrow x = x$.

Therefore, $\mathcal{I}^*$ is precisely the quasivariety of all pseudo-BCI-algebras. □

4. Conclusion

The pseudo-BCI-logic is a non-commutative version of the BCI-logic – it has two different implications $\to$ and $\rightsquigarrow$. In order to be algebraizable we have to extend it on one inference rule (Imp). This leads us to formulate and prove the main result of the paper that pseudo-BCI-algebras are an algebraic counterpart of this extended logic (Theorem 3.2). We think this logic is so close to original one that it is worth studying its algebraic models – pseudo-BCI-algebras.

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