

Josep Maria Font

## ATOMS IN A LATTICE OF THEORIES

### Abstract

The logic *Prot* is defined, in a language with just the implication connective  $\rightarrow$ , by the axiom of *reflexivity* or *identity* “ $\varphi \rightarrow \varphi$ ” and the rule of *Modus Ponens* “from  $\varphi$  and  $\varphi \rightarrow \psi$  to infer  $\psi$ ”; its theorems are formulas of the form: “ $\varphi \rightarrow \varphi$ ”. This paper continues the study of this logic as begun in the previous paper “The simplest protoalgebraic logic”. Here, I study some points of the lattice structure of its set of theories, which shows some unusual features. I prove it is atomic, determine its (denumerable) atoms, prove that there is a single atom below any principal theory, and prove that principal theories form a sub-semilattice of the lattice of theories which is order-isomorphic to the tree of finite sequences of natural numbers. I also prove that this logic has a non-term-definable weak disjunction operation.

The logic *Prot* is the logic in the language  $\langle \rightarrow \rangle$  of type  $\langle 2 \rangle$  axiomatized by

the axiom  $x \rightarrow x$  (R)

and the rule  $x, x \rightarrow y / y$ . (MP)

While *Prot* is the *name* of this logic, I will use  $\mathcal{P}$  as its *symbol* in mathematical expressions. Thus, the consequence relation of this logic will be denoted by  $\vdash_{\mathcal{P}}$ , and the associated closure operator by  $C_{\mathcal{P}}$ , so that the theory generated by a set  $\Gamma$  of formulas will be denoted by  $C_{\mathcal{P}}\Gamma$ . I will write  $\vdash_{\mathcal{P}} \varphi$  instead of  $\emptyset \vdash_{\mathcal{P}} \varphi$ .

The properties of implication included in the definition of *Prot* are arguably prototypical properties of an implication connective that are found

everywhere such a connective is considered. They are, however, very basic or minimal, and it might seem that very little can be deduced from them. Still, I think the properties of this logic are worth studying, even if the conclusion of such a study may be that “this logic has almost no properties”. Such a study, begun in [4], is a kind of revision of the main notions and techniques in abstract algebraic logic [3, 5, 6], and it may lead to a reflection on the (lack of) strength of the laws of *Identity* and *Modus Ponens* when they are not accompanied by other properties.

The following relevant facts were established in [4]:  $\mathcal{P}rot$  is the simplest protoalgebraic logic [2], it is not equivalential, weakly algebraizable or selfextensional; it does not have an algebraic semantics [1]; it has neither a conjunction nor a disjunction, or any kind of deduction-detachment theorem besides the local and parameterized deduction-detachment theorem that all protoalgebraic logics satisfy. Presumably this logic can serve as a counterexample to many points in abstract algebraic logic; in [4] it was used to establish that the local and parameterized deduction-detachment theorem does not imply its non-local version, something that had been lacking in the literature.

In this paper I study the atomic structure of the lattice of theories of  $\mathcal{P}rot$  and the structure of the set of principal theories, which is shown to be a meet sub-semilattice of the lattice of theories and to be order-isomorphic to the tree of finite sequences of natural numbers ordered under the extension relation. Other unusual features of  $\mathcal{P}rot$  arise as well.

#### NOTATIONS 1.

The set of atomic formulas or **variables** is denoted by  $Var$ ; the letters  $x, y, z$  will denote variables.

The **algebra of formulas** is denoted by  $\mathbf{Fm} = \langle Fm, \rightarrow \rangle$ , where  $Fm$  denotes the set of all formulas of this similarity type. The letters  $\alpha, \beta, \gamma, \delta, \varphi, \psi$ , perhaps with subindices, will denote formulas, and the letters  $\Gamma, \Delta$ , perhaps with subindices as well, will denote sets of formulas.

Let  $\vec{\alpha}$  denote a finite sequence of formulas and  $\beta$  a formula. Then  $\vec{\alpha} \rightarrow \beta$  denotes the nested formula  $\alpha_1 \rightarrow (\alpha_2 \rightarrow (\alpha_3 \rightarrow \dots (\alpha_n \rightarrow \beta) \dots))$  when  $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_n \rangle$  for some  $n \geq 1$ , and denotes the formula  $\beta$  when  $\vec{\alpha}$  is the empty sequence; formally, this is (easily) defined by induction (but I think this description is clear enough).

A formula  $\beta$  is a **final subformula** of a formula  $\varphi$  when  $\varphi = \vec{\alpha} \rightarrow \beta$  for some finite sequence  $\vec{\alpha}$  of formulas; this sequence is called the **prefix** (of  $\beta$  in  $\varphi$ ). A **final variable** of a formula is a final subformula that is a variable.

If  $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_n \rangle$  is a finite, non-empty sequence of formulas, then its final segments are denoted as  $\vec{\alpha}_{[k]} := \langle \alpha_k, \dots, \alpha_n \rangle$ ; we also need to set  $\vec{\alpha}_{[n+1]} := \emptyset$  for convenience.

Note that, according to the definition of  $\vec{\alpha} \rightarrow \beta$ , any formula is a final subformula of itself (with the empty prefix). A formula may have several final subformulas, but it always has one and only one final variable. Every formula can be written in the form  $\vec{\alpha} \rightarrow x$  in a *unique* way, where  $x$  is its final variable, and then its final subformulas have the form  $\vec{\alpha}_{[k]} \rightarrow x$  for  $1 \leq k \leq n + 1$ , where  $n$  is the length of  $\vec{\alpha}$ .

The following results from [4] (which are easy to prove directly) will be used here. First, the theorems of this logic and the consequences of a single assumption can be exactly determined:

PROPOSITION 1.  $\vdash_{\mathcal{P}} \varphi$  if and only if  $\varphi = \alpha \rightarrow \alpha$  for some  $\alpha$ .  $\square$

PROPOSITION 2.  $\alpha \vdash_{\mathcal{P}} \beta$  if and only if  $\beta$  satisfies one of the following conditions:

- (a)  $\beta$  is a theorem.
- (b)  $\beta = \alpha$ .
- (c)  $\alpha = \vec{\delta} \rightarrow \beta$  for some finite, non-empty sequence  $\vec{\delta}$  of theorems; that is,  $\beta$  is a final subformula of  $\alpha$  with a prefix of theorems.  $\square$

Using this property, it is possible to construct *all* formulas that imply a given non-theorem  $\beta$ : they are all the formulas of the form  $\vec{\delta} \rightarrow \beta$  for some sequence  $\vec{\delta}$  of theorems. Conversely, it is possible to find all the formulas that follow from a given formula  $\alpha$  alone. The structure of these sets will be described in Theorem 15.

COROLLARY 3.  $\alpha \dashv\vdash_{\mathcal{P}} \beta$  if and only if  $\alpha$  and  $\beta$  are both theorems, or  $\alpha = \beta$ .  $\square$

Consequences of larger sets of assumptions can be more complicated and there is no similar characterization, but still the following properties will be useful:

PROPOSITION 4. *If  $\Gamma \vdash_{\mathcal{P}} \beta$ , then  $\beta$  is a theorem or  $\beta$  is a final subformula of some formula in  $\Gamma$ .*  $\square$

Note that in particular, the second possibility implies that the variables in  $\beta$  appear in some formula in  $\Gamma$ .

COROLLARIES 5.

1. *If  $\Gamma$  is a set of variables, then  $C_{\mathcal{P}}\Gamma = \Gamma \cup C_{\mathcal{P}}\emptyset$ .*
2. *A finitely generated theory contains only a finite number of non-theorems.*
3. *No finite set is inconsistent; in particular there does not exist an inconsistent formula.*  $\square$

The set of theories of  $\mathcal{P}rot$  is a complete lattice, as in all logics. Its bottom element is the set  $C_{\mathcal{P}}\emptyset$  of theorems of  $\mathcal{P}rot$ , and its top element is the set  $Fm$  of all formulas. It was shown in [4] that this lattice is *non-distributive*. It is easy to see that, in general, all atoms in a lattice of theories of a logic  $\mathcal{L}$  (if there are any) must be principal theories, i.e., theories of the form  $C_{\mathcal{L}}\{\varphi\}$  for some formula  $\varphi$ .

In order to reduce the burden in terminology and notation it will be convenient to say, except in the statements of the results, that a formula  $\varphi$  **is an atom** when  $C_{\mathcal{P}}\{\varphi\}$  is an atom in the lattice of theories (this must not be confused with “atomic” formulas, the alternative name for variables; actually, we will see that all variables are atoms in this sense but not conversely). The reason for this is that principal theories (excluding the set of theorems) are in one-to-one correspondence with their generators (this is just a restatement of Corollary 3):

LEMMA 6. *If  $\alpha$  and  $\beta$  are non-theorems, then  $C_{\mathcal{P}}\{\alpha\} \neq C_{\mathcal{P}}\{\beta\}$  if and only if  $\alpha \neq \beta$ .*  $\square$

It will also be convenient to say that a formula  $\varphi$  **is below** a set  $\Gamma$  or a formula  $\psi$  when  $C_{\mathcal{P}}\{\varphi\} \subseteq C_{\mathcal{P}}\Gamma$  or  $C_{\mathcal{P}}\{\varphi\} \subseteq C_{\mathcal{P}}\{\psi\}$ , i.e., when  $\Gamma \vdash_{\mathcal{P}} \varphi$  or  $\psi \vdash_{\mathcal{P}} \varphi$ , respectively; in these situations,  $\Gamma$  or  $\psi$  are **above**  $\varphi$ .

Observe that from (MP) it follows that:

LEMMA 7. *If  $\vec{\delta}$  is a finite sequence of theorems and  $\gamma$  is not a theorem, then  $\vec{\delta} \rightarrow \gamma$  is not a theorem.*  $\square$

Let us call **almost closed** a set of formulas  $\Gamma$  such that  $\Gamma \cap C_{\mathcal{P}}\emptyset = \emptyset$  and  $C_{\mathcal{P}}\Gamma = \Gamma \cup C_{\mathcal{P}}\emptyset$ ; that is, a set of non-theorems such that in order to close it one needs to add just the theorems to it.

Trivially, in any logic, removing the theorems from an arbitrary theory will produce an almost closed set, but in general these will be rather large sets. The surprising feature of  $\mathcal{P}rot$  is that it has many very small or very simple almost closed sets: by Corollary 5, item 5, any set of variables is almost closed, and by item 5, many finite sets are almost closed as well, namely those produced by removing theorems from finitely generated theories. The results that I detail now characterize all one-element almost closed sets.

PROPOSITION 8. *A one-element set  $\{\varphi\}$  is almost closed if and only if the principal theory  $C_{\mathcal{P}}\{\varphi\}$  is an atom of the lattice of theories of  $\mathcal{P}rot$ .*

PROOF: If  $C_{\mathcal{P}}\{\varphi\} = \{\varphi\} \cup C_{\mathcal{P}}\emptyset$  there can be no set of formulas strictly between  $C_{\mathcal{P}}\emptyset$  and  $C_{\mathcal{P}}\{\varphi\}$ ; this shows that  $C_{\mathcal{P}}\{\varphi\}$  is an atom. Conversely, assume that  $C_{\mathcal{P}}\{\varphi\}$  is an atom and take any  $\psi \in C_{\mathcal{P}}\{\varphi\}$  such that  $\psi \notin C_{\mathcal{P}}\emptyset$ . Then  $C_{\mathcal{P}}\emptyset \subsetneq C_{\mathcal{P}}\{\psi\} \subseteq C_{\mathcal{P}}\{\varphi\}$ , and since  $C_{\mathcal{P}}\{\varphi\}$  is assumed to be an atom,  $C_{\mathcal{P}}\{\psi\} = C_{\mathcal{P}}\{\varphi\}$ . But by Lemma 6, this implies that  $\psi = \varphi$ . Therefore  $C_{\mathcal{P}}\{\varphi\} = \{\varphi\} \cup C_{\mathcal{P}}\emptyset$ , that is,  $\{\varphi\}$  is almost closed.  $\square$

THEOREM 9. *Let  $\varphi$  be a formula of any of the following kinds:*

1.  $\varphi$  is a variable.
2.  $\varphi = \alpha \rightarrow \beta$  with  $\alpha$  not a theorem and  $\alpha \neq \beta$ .
3.  $\varphi = \alpha \rightarrow \beta$  with  $\beta$  a theorem and  $\alpha \neq \beta$ .

*Then  $\{\varphi\}$  is almost closed, and therefore  $C_{\mathcal{P}}\{\varphi\}$  is an atom of the lattice of theories of  $\mathcal{P}rot$ .*

PROOF: Observe that by Proposition 1, none of the presented formulas is a theorem. In cases 1 and 2, the fact that  $\{\varphi\}$  is almost closed is a straightforward consequence of Proposition 2. In case 3, I will now show

that the set  $\Gamma := \{\varphi\} \cup C_{\mathcal{P}}\emptyset$  is closed under (MP): Assume  $\gamma, \gamma \rightarrow \delta \in \Gamma$ . If  $\gamma \rightarrow \delta \in C_{\mathcal{P}}\emptyset$ , then  $\gamma = \delta$ ; therefore  $\delta \in \Gamma$ . Otherwise,  $\gamma \rightarrow \delta = \varphi = \alpha \rightarrow \beta$ , i.e.,  $\gamma = \alpha$  and  $\beta = \delta$ , and by the assumption on  $\varphi$  this implies that  $\delta$  is a theorem, therefore  $\delta \in \Gamma$  as well. This shows that  $C_{\mathcal{P}}\{\varphi\} = C_{\mathcal{P}}\Gamma = \Gamma = \{\varphi\} \cup C_{\mathcal{P}}\emptyset$ , that is,  $\{\varphi\}$  is almost closed.  $\square$

**THEOREM 10.** *The lattice of theories of  $\mathcal{P}rot$  is atomic, and the atoms are exactly all the theories described in Theorem 9.*

**PROOF:** Let  $\Gamma$  be any theory of  $\mathcal{P}rot$  with  $C_{\mathcal{P}}\emptyset \subsetneq \Gamma$ . I will show that  $\Gamma$  must contain a formula  $\varphi$  of one of the kinds in Theorem 9; thus  $C_{\mathcal{P}}\{\varphi\} \subseteq \Gamma$  and hence the lattice is atomic and there will be no other atoms than the theories of this form. Note that  $\varphi \notin C_{\mathcal{P}}\emptyset$  for all these  $\varphi$ . To reason by contradiction, let us consider the set  $\Gamma \setminus C_{\mathcal{P}}\emptyset$  and assume it contains no formula of any of the three types mentioned. Then all its formulas must be of the form  $\alpha \rightarrow \beta$  with  $\alpha$  a theorem and  $\beta$  not a theorem. Let  $\alpha \rightarrow \beta$  be a formula of  $\Gamma \setminus C_{\mathcal{P}}\emptyset$  of least complexity, which must obviously exist. Since  $\alpha \rightarrow \beta \in \Gamma$  and  $\alpha \in C_{\mathcal{P}}\emptyset \subseteq \Gamma$ , it follows that  $\beta \in \Gamma$ . But  $\beta \notin C_{\mathcal{P}}\emptyset$  therefore  $\beta \in \Gamma \setminus C_{\mathcal{P}}\emptyset$  and  $\beta$  has less complexity than  $\alpha \rightarrow \beta$ : a contradiction.  $\square$

Observe that kinds 2 and 3 are not mutually exclusive, and that formulas  $\alpha \rightarrow \beta$ , where  $\alpha$  and  $\beta$  are two different theorems, are indeed atoms (of kind 3). Principal theories other than atoms or the theorems correspond to formulas of the form  $\alpha \rightarrow \beta$  with  $\alpha$  a theorem and  $\beta$  a non-theorem. Besides, all the non-principal theories are also non-atoms.

If  $\Gamma$  is an arbitrary theory, then the atoms below it are the formulas in  $\Gamma \setminus C_{\mathcal{P}}\emptyset$  of one of the forms in Theorem 9. However, this is not very informative and, worse, the proof of Theorem 10 is non-constructive. In the case of principal theories one can obtain a more precise result:

**THEOREM 11.** *If  $\alpha$  is a theorem and  $\beta$  is not, then the non-atom  $C_{\mathcal{P}}\{\alpha \rightarrow \beta\}$  has a single atom below it, which can be effectively obtained by inspection of  $\alpha \rightarrow \beta$ .*

**PROOF:** Write  $\alpha \rightarrow \beta = \vec{\delta} \rightarrow x$  in unique form, with  $x$  a variable and  $\vec{\delta}$  of length  $n \geq 1$ ; thus  $\delta_1 = \alpha$  is a theorem while  $\beta = \vec{\delta}_{[2]} \rightarrow x$  is not a theorem. By Proposition 2, the non-theorems that are below  $\alpha \rightarrow \beta$  are those of the form  $\vec{\delta}_{[k]} \rightarrow x$  such that  $\delta_1, \dots, \delta_{k-1}$  are theorems. I now show that there

is exactly one atom among these. There are three cases, corresponding to the three kinds of atoms described in Theorem 9:

1) If all  $\delta_i$  are theorems, then the variable  $x$  is the wanted atom, and by Lemma 7 all the other final subformulas are of the form  $\delta_k \rightarrow (\vec{\delta}_{[k+1]} \rightarrow x)$  with  $\delta_k$  a theorem and  $\vec{\delta}_{[k+1]} \rightarrow x$  not a theorem, hence they are not atoms.

Otherwise, there is some non-theorem in  $\vec{\delta}$ . Let  $k$  be the smallest  $i$  such that  $\delta_i$  is not a theorem; i.e.,  $\delta_i$  is a theorem for all  $i < k$ , and  $\delta_k$  is not a theorem. Then we distinguish two cases which will produce atoms of the other two kinds in Theorem 9:

2) If  $\delta_k \neq \vec{\delta}_{[k+1]} \rightarrow x$ , then  $\vec{\delta}_{[k]} \rightarrow x = \delta_k \rightarrow (\vec{\delta}_{[k+1]} \rightarrow x)$  is not a theorem and therefore it is an atom of the second kind below  $\alpha \rightarrow \beta$ . Using Lemma 7 as before, we see that all longer final subformulas of  $\alpha \rightarrow \beta$  are not atoms, while the shorter ones do not follow from it by Proposition 2, therefore they are not below it (while they may be atoms themselves or not).

3) If  $\delta_k = \vec{\delta}_{[k+1]} \rightarrow x$ , then let  $j$  be the largest  $i < k$  such that  $\delta_i \neq \vec{\delta}_{[i+1]} \rightarrow x$ . Thus,  $j < k$ , and for  $j < i \leq k$ ,  $\vec{\delta}_{[i]} \rightarrow x = \delta_i \rightarrow (\vec{\delta}_{[i+1]} \rightarrow x)$  are theorems, hence they are not atoms. Moreover, since  $j < k$ ,  $\delta_j$  is a theorem and  $\vec{\delta}_{[j]} \rightarrow x = \delta_j \rightarrow (\vec{\delta}_{[j+1]} \rightarrow x)$  is an atom of the third kind below  $\alpha \rightarrow \beta$ . The same reasoning as before shows that the longer final subformulas are not atoms, while the shorter ones either are not atoms or do not follow from  $\alpha \rightarrow \beta$ .

Thus, in each case a single atom below  $\alpha \rightarrow \beta$  has been determined.  $\square$

As observed in the proof, the atom found is a final subformula of  $\alpha \rightarrow \beta$ ; the proof actually describes an algorithm to find it. As examples of the three different situations in the proof, consider the following ( $\alpha, \beta, \gamma$  are arbitrary formulas):

- The atom below the formula  $(\alpha \rightarrow \alpha) \rightarrow ((\beta \rightarrow \beta) \rightarrow x)$  is the variable  $x$ .
- The atom below the formula

$$(\alpha \rightarrow \alpha) \rightarrow ((\gamma \rightarrow \gamma) \rightarrow ((\beta \rightarrow \beta) \rightarrow x) \rightarrow y))$$

is the formula

$$((\beta \rightarrow \beta) \rightarrow x) \rightarrow y.$$

- The atom below the formula

$$(\alpha \rightarrow \alpha) \rightarrow ((\gamma \rightarrow \gamma) \rightarrow (((\beta \rightarrow \beta) \rightarrow x) \rightarrow ((\beta \rightarrow \beta) \rightarrow x)))$$

is the formula

$$(\gamma \rightarrow \gamma) \rightarrow (((\beta \rightarrow \beta) \rightarrow x) \rightarrow ((\beta \rightarrow \beta) \rightarrow x)).$$

Thus, if  $\varphi = \vec{\delta} \rightarrow \alpha$  with the  $\delta_i$  theorems and  $\alpha$  is the atom below  $\varphi$ , then the formulas that are below  $\varphi$  are exactly  $\alpha$  and the final subformulas of  $\varphi$  longer than  $\alpha$ , and they form a chain

$$\varphi \vdash_{\mathcal{P}} \vec{\delta}_{[2]} \rightarrow \alpha \vdash_{\mathcal{P}} \dots \vdash_{\mathcal{P}} \delta_{n-1} \rightarrow (\delta_n \rightarrow \alpha) \vdash_{\mathcal{P}} \delta_n \rightarrow \alpha \vdash_{\mathcal{P}} \alpha \quad (1)$$

which constitutes the set of all non-theorems following from  $\varphi$ .

COROLLARIES 12.

1. Every principal theory different from  $C_{\mathcal{P}}\emptyset$  is either an atom or has a single atom below it.
2. Every non-theorem can be written in the form  $\vec{\delta} \rightarrow \alpha$  with the  $\delta_i$  theorems and  $\alpha$  an atom, and in a unique way (including the empty  $\vec{\delta}$  for atoms).
3. There is a finite chain of non-atoms between a non-atom and the (unique) atom below it.  $\square$

THEOREM 13. *The meet of any two principal theories is a principal theory.*

PROOF: Let  $C_{\mathcal{P}}\{\varphi_1\}$  and  $C_{\mathcal{P}}\{\varphi_2\}$  be any two principal theories (the trivial case of the set of theorems is excluded). We distinguish two cases:

1) If  $\varphi_1$  and  $\varphi_2$  are above the same atom  $\alpha$ , then by Corollary 12.12 there are two sequences of theorems  $\vec{\delta}_1$  and  $\vec{\delta}_2$  such that  $\varphi_i = \vec{\delta}_i \rightarrow \alpha$  for  $i = 1, 2$ . We again distinguish a number of cases:

1.1) If  $\vec{\delta}_1$  is a final segment of  $\vec{\delta}_2$ , then  $\varphi_1$  is a final subformula of  $\varphi_2$  and its prefix is made of theorems; therefore  $\varphi_2 \vdash_{\mathcal{P}} \varphi_1$  and hence  $C_{\mathcal{P}}\{\varphi_1\} \cap C_{\mathcal{P}}\{\varphi_2\} = C_{\mathcal{P}}\{\varphi_2\}$ .

1.2) If  $\vec{\delta}_2$  is a final segment of  $\vec{\delta}_1$ , the dual holds.

1.3) If neither sequence of theorems is a final segment of the other but they have a common final segment  $\vec{\delta}_3$ , then clearly  $C_{\mathcal{P}}\{\varphi_1\} \cap C_{\mathcal{P}}\{\varphi_2\} = C_{\mathcal{P}}\{\varphi_3\}$  for  $\varphi_3 := \vec{\delta}_3 \rightarrow \alpha$ .

1.4) If they have no common final segment, then clearly  $C_{\mathcal{P}}\{\varphi_1\} \cap C_{\mathcal{P}}\{\varphi_2\} = C_{\mathcal{P}}\{\alpha\}$ .

2) If  $\varphi_1$  and  $\varphi_2$  are above different atoms  $\alpha_1$  and  $\alpha_2$ , so that  $\varphi_i = \vec{\delta}_i \rightarrow \alpha_i$  for  $i = 1, 2$ , it is clear that these two formulas cannot have a common final subformula that includes the atoms. Therefore,  $C_{\mathcal{P}}\{\varphi_1\} \cap C_{\mathcal{P}}\{\varphi_2\} = C_{\mathcal{P}}\emptyset$ , which is also a principal theory.  $\square$

Observe that the proof is constructive, i.e., it works by direct inspection of the grammatical structure of the formulas involved. The property in Theorem 13 holds in all logics that are *disjunctive in the weak sense*, that is, logics having a binary formula, denoted by  $x \vee y$ , such that for all formulas  $\varphi_1, \varphi_2$ ,  $C_{\mathcal{L}}\{\varphi_1\} \cap C_{\mathcal{L}}\{\varphi_2\} = C_{\mathcal{L}}\{\varphi_1 \vee \varphi_2\}$ . It was proved in [4] that no such connective exists for *Prot*. Thus, what we obtain here is a non-term-definable function  $\mathfrak{f}$  (uniquely defined if a theorem is fixed for case 2 of the above proof) which acts in a similar way to a disjunction in the weak sense; that is, for every two formulas  $\varphi_1$  and  $\varphi_2$  it holds that  $C_{\mathcal{P}}\{\varphi_1\} \cap C_{\mathcal{P}}\{\varphi_2\} = C_{\mathcal{P}}\{\mathfrak{f}(\varphi_1, \varphi_2)\}$ .

COROLLARY 14. *Principal theories form a meet sub-semilattice of the lattice of theories.*  $\square$

The order structure of this semilattice can be completely determined:

THEOREM 15. *For each principal theory there is a denumerable tree of principal theories above it, all having below them the same atom as the given principal theory. This tree is order-isomorphic to the tree  $\omega^{<\omega}$  of finite sequences of natural numbers ordered by extension.*

PROOF: Let  $\varphi$  be a generator of the principal theory under scrutiny, and let  $\langle \alpha_n : n \geq 1 \rangle$  be a fixed enumeration of all the theorems of *Prot*. For each finite sequence  $s$  of (not necessarily distinct) natural numbers let  $\vec{\delta}_s := \langle \alpha_{s(n)}, \dots, \alpha_{s(1)} \rangle$ , where  $n$  is the length of  $s$  (for notational convenience here  $\omega$  is taken to begin with 1); let also  $\vec{\delta}_\emptyset := \emptyset$ . Then the set of theories  $\{C_{\mathcal{P}}\{\vec{\delta}_s \rightarrow \varphi\} : s \in \omega^{<\omega}\}$  is the requested set. If  $s, t \in \omega^{<\omega}$  it is clear that  $C_{\mathcal{P}}\{\vec{\delta}_t \rightarrow \varphi\} \subseteq C_{\mathcal{P}}\{\vec{\delta}_s \rightarrow \varphi\}$  if and only if  $\vec{\delta}_s \rightarrow \varphi \vdash_{\mathcal{P}} \vec{\delta}_t \rightarrow \varphi$  if and only if  $\vec{\delta}_t \rightarrow \varphi$  is a final subformula of  $\vec{\delta}_s \rightarrow \varphi$ , and by the way the formulas  $\vec{\delta}_s \rightarrow \varphi$  are defined this happens if and only if  $t$  is an initial segment of

$s$  (i.e.,  $s$  is an extension of  $t$ ). Since two distinct formulas of this form are never interderivable and they are never theorems, this establishes the wanted order-isomorphism.  $\square$

In this tree, every node has a denumerable number of successors. The bottom of the lattice, the set of theorems (which is indeed a principal theory, generated by any of its members), also has a denumerable number of successors (the atoms). Therefore:

COROLLARY 16. *The meet-semilattice of all principal theories of  $\text{Prot}$  is order-isomorphic to the tree  $\omega^{<\omega}$  of finite sequences of natural numbers ordered by extension, and its root is the set of theorems.*  $\square$

Given any point (non-root) in this tree, the finite branch that connects it to the root is the chain (1) of its proper consequences (i.e., non-theorems).

Recall that a lattice is *atomistic* when all its elements are joins of atoms (indeed, every element is the join of all atoms below it). Two consequences of Theorem 15 are:

COROLLARY 17. *The lattice of theories of  $\text{Prot}$  is not atomistic.*  $\square$

COROLLARY 18. *No principal theory of  $\text{Prot}$  is maximal.*  $\square$

By Corollary 5, item 5, there are no inconsistent finitely-generated theories. But there are non-trivial inconsistent theories:

PROPOSITION 19. *The set  $At := \{\varphi \in Fm : C_{\mathcal{P}}\{\varphi\} \text{ is an atom}\}$  is an inconsistent set of formulas.*

PROOF: It is enough to prove that if a formula  $\varphi$  is not a theorem, then it follows from  $At$ . Take any variable  $x$  (different from  $\varphi$  if  $\varphi$  is a variable). By Theorem 9, both  $x$  and  $x \rightarrow \varphi$  are atoms, and clearly  $\varphi \in C_{\mathcal{P}}\{x, x \rightarrow \varphi\} \subseteq C_{\mathcal{P}}At$ .  $\square$

Contrary to what might seem natural, the set  $At$  of all atoms is not a *minimal* inconsistent set: while no atom follows from a single atom, an atom may follow from a set of other atoms; as an example, the same reasoning in the previous proof shows that a chosen, arbitrary atom can be removed from  $At$  while preserving consequence, and hence inconsistency.

ACKNOWLEDGEMENTS. I am grateful to the referees, and acknowledge in particular the suggestions of one of them, which certainly improved the paper. While writing this paper I was partially funded by the research project MTM2011-25747 from the government of Spain, which includes FEDER funds from the European Union, and the research grant 2009SGR-1433 from the government of Catalonia.

## References

- [1] W. Blok and J. Rebagliato, *Algebraic semantics for deductive systems*, **Studia Logica (Special issue on Abstract Algebraic Logic, Part II)** 74 (2003), pp. 153–180.
- [2] J. Czelakowski, **Protoalgebraic logics**, Vol. 10 of **Trends in Logic – Studia Logica Library**, Kluwer Academic Publishers, Dordrecht, 2001.
- [3] J. M. Font, *Beyond Rasiowa’s algebraic approach to non-classical logics*, **Studia Logica** 82 (2006), pp. 172–209.
- [4] J. M. Font, *The simplest protoalgebraic logic*, **Mathematical Logic Quarterly**, to appear.
- [5] J. M. Font and R. Jansana, **A general algebraic semantics for sentential logics. Second revised edition**. Vol. 7 of **Lecture Notes in Logic**, Association for Symbolic Logic, 2009. Electronic version freely available through Project Euclid at <http://projecteuclid.org/euclid.lnl/1235416965>.
- [6] J. M. Font, R. Jansana and D. Pigozzi, *A survey of abstract algebraic logic*, **Studia Logica (Special issue on Abstract Algebraic Logic, Part II)** 74 (2003), pp. 13–97. With an update in vol. 91 (2009), pp. 125–130.

Departament de Probabilitat, Lògica i Estadística  
 Universitat de Barcelona (UB)  
 Gran Via de les Corts Catalanes 585, E-08007 Barcelona, Spain  
 e-mail: jmfont@ub.edu