Bulletin of the **S**ection of **L**ogic
Volume 42:1/2 (2013), pp. 53–67

Marek Nowak

ON SOME APPLICATION OF THE CONCEPT OF RESIDUATED PAIR OF MAPPINGS

Abstract

The paper presents one of the criteria for a complete sublattice of a given complete lattice, namely: the existence of so-called *interior-closure pair of associated operations on a given subset of the lattice*. It is shown that this pair coincides with such a residuated pair of mappings that is identical with its pair of interior-closure operations. All the interior-closure pairs of associated operations on a subset B of the family $P(A)$ of all the subsets of given set A are considered. Moreover, the proposal of a "natural" generalization of the concept of rough sets, similar to that of [10], based on the notion of interior-closure pair of associated operations on a set of discourse, is provided.

1. An interior-closure pair of associated operations on a given set

Let a nonempty poset $(A, \leq)$ and a nonempty set $B \subseteq A$ be such that for any $a \in A$ there is the greatest element in the set $\{x \in B : x \leq a\}$ and, simultaneously, there is the least one in the set $\{x \in B : a \leq x\}$. The mapping $I : A \longrightarrow B$ which assigns to each $a \in A$ the greatest element from $\{x \in B : a \leq x\}$ is obviously an *interior* operation, i.e., it fulfils the conditions:

$Ia \leq a,$
$a \leq b \Rightarrow Ia \leq Ib,$
$Ia \leq IIa.$

While the mapping $C : A \longrightarrow B$ which associates with each $a \in A$ the least element from $\{x \in B : a \leq x\}$ turns out to be a closure operation, i.e., it meets the conditions:

$a \leq Ca$,
$a \leq b \Rightarrow Ca \leq Cb$,
$CCa \leq Ca$.

Notice that the sets of all open elements and of all closed elements are equal: $\{a \in A : a = Ia\} = \{Ia : a \in A\} = \{a \in A : a = Ca\} = \{Ca : a \in A\} = B$.

On the other hand, starting from any nonempty poset $(A, \leq)$ with the operations of interior $I : A \longrightarrow A$, and closure $C : A \longrightarrow A$ such that $I[A] = C[A]$, we obtain that for any $a \in A$, Ia is the greatest element in $\{x \in I[A] : x \leq a\}$ and Ca is the least one in $\{x \in C[A] : a \leq x\}$.

Hereafter, such a pair (I, C) will be called an *interior-closure pair of associated operations on the set* $B = I[A] = C[A]$. The inspiration to formulate such a notion is taken from [5].

Given a poset $(A, \leq)$, the most trivial example of interior-closure pair of associated operations on its subset is the pair (Id_A, Id_A) of identity functions of the set A associated on it.

Any nonempty subset $B \subseteq A$ which forms a complete sublattice of a complete lattice $(A, \leq)$, leads to the following definition of interior-closure pair of associated operations on B, for any $a \in A$:

(*) $Ia = \sup\{x \in B : x \leq a\}$,
$Ca = \inf\{x \in B : a \leq x\}$.

Conversely, when in a complete lattice $(A, \leq)$ there is an interior-closure pair of associated operations on a subset $B \subseteq A$, then $(B, \leq)$ is a complete sublattice of $(A, \leq)$:

FACT. *Let* $(A, \leq)$ *be a complete lattice and* $\emptyset \neq B \subseteq A$. *The following conditions are equivalent:*
(i) $(B, \leq)$ *is a complete sublattice of* $(A, \leq)$,
(ii) *there exists an interior-closure pair of associated operations on* B.

PROOF. $(i) \Rightarrow (ii)$: It is obvious that the definitions (*) determine an interior-closure pair of associated operations on B.

$(ii) \Rightarrow (i)$: Suppose (ii) and let $B' \subseteq B$. Then there exists the least element, let us say, d in $D = \{x \in B : \inf B' \leq x\}$. So $\inf B' \leq d$. Moreover, $d = \inf D$ and $B' \subseteq D$. Hence $d \leq \inf B'$. Therefore, $d = \inf B'$. Since $d \in B$, so $\inf B' \in B$. Analogously for the least upper bound of B'. □

EXAMPLE 1. Given a complete lattice $(A, \leq)$ consider the complete lattice $(A^A, \leq)$ of all the mappings from A into A ordered in the natural way: $f \leq g$ iff $f(x) \leq g(x)$, for all $x \in A$. Then the poset $(Mon, \leq)$ of all *monotone* functions from A into A is a complete sublattice of the lattice $(A^A, \leq)$ ($f \in A^A$ is *monotone* iff $f(x) \leq f(y)$ whenever $x \leq y$, for all $x, y \in A$). Recall that given any $M \subseteq Mon$ and $x \in A$ we have: $(\inf M)(x) = \inf\{f(x) : f \in M\}$, $(\sup M)(x) = \sup\{f(x) : f \in M\}$.

In this way, we obtain an interior-closure pair of operations I, C associated on Mon: for any $f : A \longrightarrow A$,

$$I(f) = \sup\{g \in Mon : g \leq f\},$$
$$C(f) = \inf\{g \in Mon : f \leq g\}.$$

2. A residuated pair of mappings

Now let us briefly recall the notion of *residuated* or *adjoint* pair of mappings. Given any posets $(A, \leq_A)$, $(B, \leq_B)$, a pair of mappings $\phi : A \longrightarrow B$, $\psi : B \longrightarrow A$ is called *residuated* iff for each $a \in A, b \in B$:

$$(res) \quad b \leq_B \phi(a) \text{ iff } \psi(b) \leq_A a,$$

or equivalently, ϕ and ψ are monotone and for any $a \in A, b \in B$:

$$\psi(\phi(a)) \leq_A a \text{ and } b \leq_B \phi(\psi(b)).$$

The term "residuated" is sometimes applied to converse orderings instead of $\leq_A$, $\leq_B$ (cf. for example [4]; in such a case, a pair (ϕ, ψ) is residuated for the posets $(A, \leq_A), (B, \leq_B)$ iff the pair (ψ, ϕ) is residuated in our sense for the posets $(B, \leq_B), (A, \leq_A)$). A function $\phi : A \longrightarrow B$ for which there exists a map $\psi : B \longrightarrow A$ such that (ϕ, ψ) is a residuated pair, is called *residuated*. Given a residuated function there is a unique ψ such that (ϕ, ψ) is a residuated pair.

For a residuated pair (ϕ, ψ) of functions, their compositions I and C of the form: $Ia = \psi(\phi(a)),\ Cb = \phi(\psi(b))$ (any $a \in A, b \in B$), are the interior and closure operations on A and B, respectively. In the sequel, we will call (I, C) a *pair of interior-closure operations designated by the residuated pair* (ϕ, ψ). As we have:

$$\{a \in A : a = Ia\} = \psi[B] \quad \text{and} \quad \{b \in B : b = Cb\} = \phi[A]$$

then the mapping ϕ restricted to the image $\psi[B]$ is a bijection from that image onto $\phi[A]$. Moreover, for any $a_1, a_2 \in \psi[B]$: $a_1 \leq_A a_2$ iff $\phi(a_1) \leq_B \phi(a_2)$, so in fact, the posets, $(\psi[B], \leq_A)$, $(\phi[A], \leq_B)$, (of all open and closed elements, respectively) are isomorphic.

When the posets $(A \leq_A)$, $(B, \leq_B)$ are complete lattices more desirable cases of residuated pairs emerge as:

(res') any mapping $\phi : A \longrightarrow B$ is residuated iff for each $A' \subseteq A$, $\phi(\inf_A A') = \inf_B \phi[A']$

(cf. for example [7]).

Example 2. The typical examples of residuated pairs employ the powersets $(P(\mathsf{K}), \subseteq), (P(\mathsf{X}), \subseteq)$ of given sets K, X and are defined with an arbitrary binary relation $\rho \subseteq \mathsf{K} \times \mathsf{X}$ in the following way: for any $K \subseteq \mathsf{K}$, $X \subseteq \mathsf{X}, \alpha \in \mathsf{X}$ and $m \in \mathsf{K}$,

$\alpha \in \phi(K)$ iff $\forall n \in \mathsf{K}(n\rho\alpha \Rightarrow n \in K)$,
$m \in \psi(X)$ iff $\exists \beta \in X,\ m\rho\beta$.

The pair (I, C) of interior-closure operations designated by the residuated pair (ϕ, ψ) is then of the form:

$m \in I(K)$ iff $\exists \alpha \in \mathsf{X}(m\rho\alpha \,\&\, \forall n \in \mathsf{K}(n\rho\alpha \Rightarrow n \in K))$ iff $\exists \alpha \in \mathsf{X}(m \in \psi(\{\alpha\}) \,\&\, \psi(\{\alpha\}) \subseteq K)$,

$\alpha \in C(X)$ iff $\forall m \in \mathsf{K}(m\rho\alpha \Rightarrow \exists \beta \in X, m\rho\beta)$ iff $\psi(\{\alpha\}) \subseteq \psi(X)$.

The definition of an exemplary residuated pair is most often taking recourse to the relation $\rho \subseteq \mathsf{K} \times \mathsf{X}$: $m\rho\alpha$ iff $m = f(\alpha)$, where $f : \mathsf{X} \longrightarrow \mathsf{K}$ is an arbitrary mapping. Then for any $K \subseteq \mathsf{K}$: $\phi(K) = \overleftarrow{f}(K)$ and for each $X \subseteq \mathsf{X}$: $\psi(X) = f[X]$. Here, the interior-closure pair (I, C) designated by (ϕ, ψ) is of the form: $I(K) = K \cap f[\mathsf{X}]$ and $C(X) = \{\alpha \in \mathsf{X} : f(\alpha) \in f[X]\}$.

So when the function f is a bijection from X onto K, we obtain that $I = Id_{P(\mathsf{K})}$, $C = Id_{P(\mathsf{X})}$.

Let us consider a simple case when $\mathsf{K} = \mathsf{X} = \{0.1\}$ and a bijection f is of the form: $f(0) = 1$, $f(1) = 0$. Then the residuated mapping $\phi : P(\{0,1\}) \longrightarrow P(\{0,1\})$, here of the form: $\phi(\emptyset) = \emptyset$, $\phi(\{0\}) = \{1\}$, $\phi(\{1\}) = \{0\}$, $\phi(\{0,1\}) = \{0,1\}$, is an isomorphism of the poset $(P(\{0,1\}), \subseteq)$ onto itself.

Consider another slightly different case when X contains more than 1-element and $a_0 \in \mathsf{X}$. Let $f : \mathsf{X} \longrightarrow \mathsf{X}$ be a constant function: $f(a) = a_0$ for each $a \in \mathsf{X}$. Then the residuated mapping $\phi : P(\mathsf{X}) \longrightarrow P(\mathsf{X})$ is defined as follows: for any $Y \subseteq \mathsf{X}$: $\phi(Y) = \mathsf{X}$ if $a_0 \in Y$, otherwise $\phi(Y) = \emptyset$. The function ϕ restricted to the set $\psi[P(\mathsf{X})] = \{\emptyset, \{a_0\}\}$ of all open elements is an isomorphism of the 2-element posets $(\{\emptyset, \{a_0\}\} \subseteq)$, $(\{\emptyset, \mathsf{X}\}, \subseteq)$.

3. A nexus between residuated pairs and interior-closure pairs of associated operations

Lemma. *Let (ϕ, ψ) be a residuated pair on a poset $(A, \leq)$ (that is $\phi, \psi \in A^A$). If ϕ is an interior operation on A, then ψ is a closure operation and the interior-closure pair designated by (ϕ, ψ) is just (ϕ, ψ) itself. Furthermore, $\phi[A] = \psi[A]$, i.e, the set of all closed elements is the same as the set of all open elements, and the appropriate isomorphism, that is ϕ restricted to the set of all open elements, is the identity function.*

Proof. Assume that (ϕ, ψ) is a residuated pair, ϕ, $\psi : A \longrightarrow A$ and the residuated function ϕ is an interior operation on A. First, we show that ψ is a closure operation. So we have, for any $a \in A$: $a \leq \phi(\psi(a)) \leq \psi(a)$. Obviously, ψ is monotone. Next notice that due to the condition (res), now in the form: $\psi(b) \leq a$ iff $b \leq \phi(a)$, any $a, b \in A$, the following conditions are equivalent: $\psi(\psi(a)) \leq \psi(a)$, $\psi(a) \leq \phi(\psi(a))$, $a \leq \phi(\phi(\psi(a)))$. However, $a \leq \phi(\psi(a)) \leq \phi(\phi(\psi(a)))$, for ϕ is an interior opration. In this way, the condition $\psi(\psi(a)) \leq \psi(a)$ follows. Thus, ψ is a closure operation on A.

Moreover, the conditions: $\phi(a) \leq \phi(\phi(a))$ and (res) imply that $\psi(\phi(a)) \leq \phi(a)$. On the other hand: $\phi(a) \leq \psi(\phi(a))$ since ψ is a closure operation on A. In that way, the interior operation I for the residuated pair (ϕ, ψ) is of the form: for any $a \in A$: $Ia = \psi(\phi(a)) = \phi(a)$, so $I = \phi$. Analogously for the closure opearation: $Ca = \phi(\psi(a)) = \psi(a)$ since $\psi(a) \leq \phi(\psi(a))$

(as equivalent to $\psi(\psi(a)) \leq \psi(a)$), and $\phi(\psi(a)) \leq \psi(a)$ (as ϕ is an interior operation).

In order to prove the last part of the lemma, observe simply that, on one hand: $\{a \in A : \phi(a) = a\} = \phi[A]$, as ϕ is an interior operation, on the other hand: $\{a \in A : \phi(a) = a\} = \psi[A]$, as in general case, the set of all open elements is exactly the counterdomain of the second mapping from the residuated pair. Obviously, ϕ restricted to the set $\{a \in A : \phi(a) = a\}$ is the identity function. □

An obvious consequence of Lemma is a statement that a residuated pair (ϕ, ψ) of mappings on a poset $(A, \leq)$ such that the residuated function ϕ is an interior operation on A, forms an interior-closure pair of operations on the set of all open elements of A.

Let us notice that when $\phi : A \longrightarrow A$ is a residuated function on a complete lattice $(A, \leq)$ and, moreover, ϕ is an interior operation on A, then from (res') it follows that ϕ is a topological interior operation that is for any $a, b \in A:\ \phi(a \wedge b) = \phi(a) \wedge \phi(b)$ and $\phi(1) = 1$, where $\wedge$ and 1 are the operation of *infimum* and the greatest element of $(A, \leq)$, respectively. In this case, the second component of residuated pair (ϕ, ψ) is a topological closure on A.

Proposition. *Let $(A, \leq)$ be a nonempty poset, $B \subseteq A$ and $\phi,\ \psi : A \longrightarrow A$ be any functions. Then the following conditions are equivalent:*

(i) (ϕ, ψ) is an interior-closure pair of associated operations on B,

(ii) (ϕ, ψ) is a residuated pair such that ϕ is an interior operation on A and $\phi[A] = B$.

Proof. $(i) \Rightarrow (ii)$: Assume (i). Then from the definiton of an interior-closure pair of associated operations on a given set we have: for any $a \in A:\ \psi(\phi(a)) = \phi(a) \leq a$ and $a \leq \psi(a) = \phi(\psi(a))$. Furthermore, ϕ, ψ are monotone. Therefore, (ϕ, ψ) is a residuated pair. The remaining facts of (ii) are obvious.

$(ii) \Rightarrow (i)$: due to Lemma. □

4. Some natural applications

First of all let us establish a form of all the interior-closure pairs of associated operations on a subset $B \subseteq P(A)$ given a poset $(P(A), \subseteq)$, where A is a set. To this aim, according to Proposition, we should consider all the mappings $\phi : (P(A), \subseteq) \longrightarrow (P(A), \subseteq)$ being residuated functions (that is in this case such that for any $\mathcal{R} \subseteq P(A)$: $\phi(\bigcap \mathcal{R}) = \bigcap \phi[\mathcal{R}]$) which are the interior operations. Let us show that such functions are exactly the interior operations of Alexandroff topological spaces. A topological space $(A, \mathcal{O})$ is called an Alexandroff space iff its topology $\mathcal{O}$ is closed on arbitrary intersections (cf. [1], [2]). So for any family $\mathcal{R}$ of open sets with respect to a residuated interior ϕ we have $\phi(\bigcap \mathcal{R}) = \bigcap\{\phi(U) : U \in \mathcal{R}\} = \bigcap\{U : U \in \mathcal{R}\} = \bigcap \mathcal{R}$, that is, $\bigcap \mathcal{R}$ is open. Conversely, an interior operation in an Alexandroff space is a residuated function: when ϕ is an interior operation of an Alexandroff topological space A, then for any nonempty family $\mathcal{R}$ of subsets of A, the set $\bigcap\{\phi(U) : U \in \mathcal{R}\}$ is open. Moreover, $\bigcap\{\phi(U) : U \in \mathcal{R}\} \subseteq \bigcap \mathcal{R}$, since $\phi(U) \subseteq U$ for each $U \in \mathcal{R}$. So $\bigcap\{\phi(U) : U \in \mathcal{R}\} \subseteq \phi(\bigcap \mathcal{R})$, for $\phi(\bigcap \mathcal{R})$ is the greatest open subset included in $\bigcap \mathcal{R}$. The converse inclusion is obvious due to monotonity of the operation ϕ. Thus we have:

the first component of an interior-closure pair of associated operations on a subset $B \subseteq P(A)$ is always an interior operation of an Alexandroff space. The set B is the family of all its open subsets.

In order to describe the form of the set B more explicitely and to present the form of the second element of the interior-closure pair, it is very convenient to remind a close nexus between the Alexandroff interior operations and *quasiorders* (reflexive and transitive binary relations) (cf. for example [10], [11]). First, we show a more general nexus between the class of all residuated mappings $\phi : (P(A), \subseteq) \longrightarrow (P(A), \subseteq)$ and the class of all binary relations defined on A (cf. for example [4]). To this aim we shall apply the notion of *Galois connection* (cf. for example [6]) which is very close to *residuated pair* (even in some papers the former replaces the latter as its equivalent).

Given any posets $(A, \leq_A)$, $(B, \leq_B)$, a pair of mappings $f : A \longrightarrow B$, $g : B \longrightarrow A$ is called a *Galois connection* iff for each $a \in A, b \in B$:

(Gl) $b \leq_B f(a)$ iff $a \leq_A g(b)$,

or equivalently, f and g are antimonotone and for any $a \in A, b \in B$:

$a \leq_A g(f(a))$ and $b \leq_B f(g(b))$.

Given a Galois connection (f, g), the compositions C_1 and C_2 of the form: $C_1(a) = g(f(a))$, $C_2(b) = f(g(b))$, any $a \in A, b \in B$, are the closure operations on A and B, respectively. They satify the following clause:

$$\{a \in A : a = C_1(a)\} = g[B] \text{ and } \{b \in B : b = C_2(b)\} = f[A].$$

In this way it follows that the mapping f restricted to the image $g[B]$ is a bijection from that image onto $f[A]$. Moreover, for any $a_1, a_2 \in g[B]$: $a_1 \leq_A a_2$ iff $f(a_2) \leq_B f(a_1)$, so in fact the posets $(g[B], \leq_A)$, $(f[A], \leq_B)$ are dually isomorphic.

Now consider two posets: $(P(A \times A), \subseteq)$, $(P(A)^{P(A)}, \leq)$ of all binary relations defined on a set A and of all the mappings from $P(A)$ into $P(A)$, respectively. Let $f: P(A \times A) \longrightarrow P(A)^{P(A)}$ be a mapping such that for any $\rho \subseteq A \times A$, $U \subseteq A$ and $y \in A$: $y \in f(\rho)(U)$ iff $(y]_\rho \subseteq U$, where $(y]_\rho = \{x \in A : x\rho y\}$. Obviously, $f(\rho)$ is, already considered (cf. Example 2), residuated function ϕ (defined by a binary relation ρ) modified by the assumption that $\mathsf{K} = \mathsf{X} = A$. Notice also that $f(\rho^\smile)$, where $\rho^\smile$ is the converse of ρ, is an operation occurring in algebraic semantics based on Kripke models for modal propositional logic. It is the operation corresponding to the necessity connective, while $\rho^\smile$ is a binary relation serving to define the truth condition in a Kripke model for a formula involving that connective (cf. for example [3]). Let $g : P(A)^{P(A)} \longrightarrow P(A \times A)$ be a mapping such that for each $\phi : P(A) \longrightarrow P(A)$ and any $x, y \in A$: $(x, y) \in g(\phi)$ iff $\forall U \subseteq A\ (y \in \phi(U) \Rightarrow x \in U)$. Then it is easy to show that the pair (f, g) is a Galois connection. Furthermore, one can show that $C_1(\rho) = \rho$ for each $\rho \subseteq A \times A$. Now let us prove the following

FACT. *For each $\phi : P(A) \longrightarrow P(A)$, $C_2(\phi) = \phi$ iff ϕ is a residuated function, that is for any nonempty $\mathcal{R} \subseteq P(A)$, $\phi(\bigcap \mathcal{R}) = \bigcap\{\phi(U) : U \in \mathcal{R}\}$ and $\phi(A) = A$.*

PROOF. Given a mapping $\phi : P(A) \longrightarrow P(A)$ by definition of closure operation C_2 we have:

(0) for each $y \in A$ and $U \subseteq A$: $y \in C_2(\phi)(U)$ iff $(y]_{g(\phi)} \subseteq U$.

($\Rightarrow$): In this direction we really show that for any relation ρ, $f(\rho)$ is a residuated function (since $C_2(\phi) = \phi$ iff $\phi \in f[P(A \times A)]$), which was expected from the moment of providing Example 2. Thus taking into account (0) let us ssume that

(1) $\forall U \subseteq A \; \forall y \in A \; ((y]_{g(\phi)} \subseteq U \text{ iff } y \in \phi(U))$.

So in particular, $\forall y \in A \; ((y]_{g(\phi)} \subseteq A \text{ iff } y \in \phi(A))$ and hence $\phi(A) = A$. Moreover, consider a nonempty $\mathcal{R} \subseteq P(A)$. Then it follows from (1) that for any $y \in A : \; y \in \phi(\bigcap \mathcal{R})$ iff $(y]_{g(\phi)} \subseteq \bigcap \mathcal{R}$ iff $\forall U \in \mathcal{R}, (y]_{g(\phi)} \subseteq U$ iff $\forall U \in \mathcal{R}, y \in \phi(U)$ iff $y \in \bigcap\{\phi(U) : U \in \mathcal{R}\}$.

($\Leftarrow$): Suppose that ϕ is residuated, so it is monotone, as well. It is sufficient to show that $C_2(\phi) \leq \phi$. Assume that this does not hold. So for some $U_0 \subseteq A : \; C_2(\phi)(U_0) \not\subseteq \phi(U_0)$. Therefore, for some $y_0 \in A$ we have:

(2) $y_0 \in C_2(\phi)(U_0)$ and

(3) $y_0 \notin \phi(U_0)$.

(2) and (0) imply that $\forall x \in A \; (x \notin U_0 \Rightarrow x \notin (y_0]_{g(\phi)})$ which by definition of the function g leads to the condition:

(4) $\forall x \in A \; (x \notin U_0 \Rightarrow \exists U \subseteq A \; (y_0 \in \phi(U) \; \& \; x \notin U))$.

From (3) and the assumption ($\phi(A) = A$) it follows that $U_0 \neq A$. Therefore, on the basis of (4), let us associate with each $x \notin U_0$ a set $U_x \subseteq A$ such that $y_0 \in \phi(U_x)$ and $x \notin U_x$. Then $y_0 \in \bigcap\{\phi(U_x) : x \notin U_0\}$. So according to the assumption we obtain that

(5) $y_0 \in \phi(\bigcap\{U_x : x \notin U_0\})$.

Moreover, the following clause holds:

(6) $\forall z \in A \; (z \notin U_0 \Rightarrow z \notin \bigcap\{U_x : x \notin U_0\})$,

since $z \notin U_z$ for $z \notin U_0$. (6) implies that $\bigcap\{U_x : x \notin U_0\} \subseteq U_0$, hence, from (5) and the assumption it follows that $y_0 \in \phi(U_0)$; a contradiction with (3). $\square$

Finally, we obtain (cf. the Theorem 1.4 of [4])

Corollary 1. *The posets* $(P(A \times A), \subseteq)$, $(Res(A), \leq)$, *where* $Res(A)$ *is the class of all residuated functions* $\phi : (P(A), \subseteq) \longrightarrow (P(A), \subseteq)$, *are dually isomorphic and the function* $f : P(A \times A) \longrightarrow Res(A)$ *(defined above) is a required dual isomorphism.*

Now taking into account Corollary 1 and the following obvious (by definition of $f(\rho)$) clauses: for each binary relation ρ on a set A:

ρ is reflexive iff $\forall U \subseteq A, \; f(\rho)(U) \subseteq U$,

ρ is transitive iff $\forall U \subseteq A, \; f(\rho)(U) \subseteq f(\rho)(f(\rho)(U))$,

one can establish

Corollary 2. *The function f restricted to the class of all quasiorders defined on a given set A is a dual isomorphism of the posets $(QOrd(A), \subseteq)$, $(ResInt(A), \leq)$, where $QOrd(A)$, $ResInt(A)$ are the classes of all quasiorders on A and of all residuated functions which are the interior operations on space A (that is the Alexandroff interior operations), respectively.*

Now consider the well-known isomorphism, let us say, f_1 from the class of all interior operations (not necessarily topological) defined on A onto the class of all the families of subsets of A which are closed on arbitrary unions (this is a dual connection to better known isomorphism between closure operations and *closure systems* (*Moore families*)): for any interior operation I on A, $f_1(I) = \{U \subseteq A : I(U) = U\} = I[P(A)]$. The converse isomorphism g_1 is of the form: for any family $\mathcal{R} \subseteq P(A)$ closed on arbitrary unions, and any $U \subseteq A$, $g_1(\mathcal{R})(U) = \bigcup\{V \in \mathcal{R} : V \subseteq U\}$. Now one can confine f_1 to the class $ResInt(A)$ and consider a dual isomorphism $f \circ f_1$ which assigns to each quasiorder ρ the family $f(\rho)[P(A)]$ closed on arbitrary intersections and unions simultaneously, i.e. the family being an Alexandroff topology.

This dual isomorphism between $(QOrd(A), \subseteq)$ and $(Alex(A), \subseteq)$, where $Alex(A)$ is the family of all Alexandroff topologies defined on A, can be reproduced from the following Galois connection (f_0, g_0). Namely, $f_0 : (P(A \times A), \subseteq) \longrightarrow (P(P(A)), \subseteq)$ is of the form: for each $U \subseteq A$, $U \in f_0(\rho)$ iff $\forall x, y \in A\ (y \in U\ \&\ x\rho y \Rightarrow x \in U)$. So f_0 associates with each binary relation ρ defined on A the family of all *downward* subsets of A with respect to ρ. The mapping $g_0 : (P(P(A)), \subseteq) \longrightarrow (P(A \times A), \subseteq)$ is defined as follows: for any family $\mathcal{R} \subseteq P(A)$ and any $x, y \in A$, $(x, y) \in g_0(\mathcal{R})$ iff $\forall U \in \mathcal{R}\ (y \in U \Rightarrow x \in U)$. So g_0 associates with any family $\mathcal{R}$ of subsets of A a quasiorder on A. Next one can obtain that f_0 restricted to $QOrd(A)$ is a dual isomorphism mapping $(QOrd(A), \subseteq)$ onto $(DW(A), \subseteq)$, where $DW(A) = \{\mathcal{R} \subseteq P(A)$: there is a quasiorder ρ such that $\mathcal{R}$ is the family of all downward sets with respect to $\rho\}$. However, $DW(A) = Alex(A)$ and $f \circ f_1 = f_0$, where both f, f_0 are restricted to the class of all quasiorders defined on A. These two equalities are the consequences of the following equivalence:

for each quasiorder ρ defined on A and any $U \subseteq A$: U is a downward set with respect to ρ iff $f(\rho)(U) = U$, i.e., U is open with respect to the Alexandroff interior operation $f(\rho)$.

This equivalence follows from the general connection between the mappings f_0 and f of the form:

for any $U \subseteq A$ and $\rho \subseteq A \times A:\ U \in f_0(\rho)$ iff $U \subseteq f(\rho)(U)$.

Finally, we can state that

a set $B \subseteq P(A)$ such that there exists an interior-closure pair of associated operations on it, has to be a family of all the downward subsets of A with respect to a quasiorder defined on A.

The function $\psi : P(A) \longrightarrow P(A)$ being the second element of the residuated pair defined by a quasiorder ρ as Example 2 shows, and, at the same time, being the second element of the interior-closure pair of associated operations on a $B \subseteq P(A)$ (due to Proposition) is of the form: for any $U \subseteq A$ and $x \in A$:

$x \in \psi(U)$ iff $\exists y \in U,\ x\rho y$.

Because for any quasiorder ρ and $U \subseteq A$, the set $\bigcup\{(y]_\rho : y \in U\}$ is downward with respect to ρ (that is, belongs to the family $f_0(\rho)$), one can show, given $x \in A,\ U \subseteq A$, that the following equivalence holds:

$\exists y \in U,\ x\rho y$ iff $\forall V \in f_0(\rho)(U \subseteq V \Rightarrow x \in V)$,

which implies that $\psi(U) = \bigcap\{V \in f_0(\rho) : U \subseteq V\}$. In this way,

the second element: ψ, of an interior-closure pair of associated operations on a given $B \subseteq P(A)$ is a closure operation corresponding to the closure system (Alexandroff topology) $f_0(\rho)$ of all the downward sets with respect to a quasiorder ρ on A. However, the closure system $f_0(\rho)$ is the family of all complements of these open sets which form the Alexandroff topology $f_0(\rho^\sim)$, where $\rho^\sim$ is the converse relation to ρ, that is the topology consisting of all the upward subsets U of A with respect to ρ (i.e. fulfilling the condition: $x \in U \ \&\ x\rho y \Rightarrow y \in U$). Therefore, ψ is the closure operation of the Alexandroff topological space whose topology is the family $f_0(\rho^\sim)$ of all the upward subsets of A with respect to the quasiorder ρ.

To conclude, let us consider some example. For a given nonempty poset $(A, \leq)$ let $\phi,\ \psi : P(A) \longrightarrow P(A)$ be a residuated pair defined by the relation $\leq$. That is, for any $U \subseteq A:\ \phi(U) = \{a \in A : (a]_\leq \subseteq U\},\ \psi(U) =$

$\bigcup\{(a]_{\leq} : a \in U\}$. So the pair of interior-closure operations designated by (ϕ, ψ) is of the form: $I = \phi$ and $C = \psi$. Obviously, in this case we have: $\{\phi(U) : U \subseteq A\} = \{\psi(U) : U \subseteq A\}$, and the counterdomains of ϕ and ψ form the family of all the downward subsets of A. In this way, given any $U \subseteq A$, $\psi(U)$ is the least downward subset containing U and $\phi(U)$ is the greatest downward subset contained in U. In other words, (ϕ, ψ) is an interior-closure pair of associated operations on the family $\mathcal{R}$ of all downward subsets of A. In different terms, the family forms a complete sublattice of the complete lattice $(P(A), \subseteq)$, i.e., it is closed on $\bigcap$ and $\bigcup$. So, according to clauses (*), for any $U \subseteq A$: $I(U) = \phi(U) = \bigcup\{V \in \mathcal{R} : V \subseteq U\}$, $C(U) = \psi(U) = \bigcap\{V \in \mathcal{R} : U \subseteq V\}$. Moreover, here I is a topological interior operation under an Alexandroff topology consisting of just all downward subsets, and C is a topological closure operation under an Alexandroff topology consisting of all the upward subsets of A.

5. Application to the rough sets

Finally, we shall generalize the Pawlak's [12] concept of rough set applying the notion of interior-closure pair of asscociated operations on a given set. First, let us briefly remind the concept. Let θ be an equivalence relation defined on a set A of discourse, conceived as so-called *indiscernibility relation* designated on A by the family of attributes (cf. [12]). Then, given $U \subseteq A$, the sets: $U_\theta = \{a \in A : [a]_\theta \subseteq U\}$ and $U^\theta = \{a \in A : [a]_\theta \cap U \neq \emptyset\}$ (where $[a]_\theta$ is the equivalence class of a) are called the *lower* and the *upper approximations* of U, respectively. Some of the important connections between the lower and upper approximations provided by Pawlak are as follows, for any $U \subseteq A$:

$(-U)_\theta = -(U^\theta)$,
$(-U)^\theta = -(U_\theta)$, where "$-$" is the sign of complement,
$(U_\theta)_\theta = (U_\theta)^\theta = U_\theta$,
$(U^\theta)^\theta = (U^\theta)_\theta = U^\theta$

(cf. the last four equalities from the list denoted as (5) in [13]). The *rough set* of the set U is identified with the equivalence class: $[U]_\equiv$ with respect to the relation $\equiv$ defined on $P(A)$ as follows: $X \equiv Y$ iff $X_\theta = Y_\theta$ and $X^\theta = Y^\theta$. Usually the rough set of U is represented by the pair (U_θ, U^θ) and the following poset of all the rough sets is considered: $(RS, \leq)$, where

$RS = \{(U_\theta, U^\theta) : U \subseteq A\}$ and for any $X, Y \subseteq A:\ (X_\theta, X^\theta) \leq (Y_\theta, Y^\theta)$ iff $(X_\theta \subseteq Y_\theta)$ and $(X^\theta \subseteq Y^\theta)$.

There are many generalizations of this concept. Nowadays the number of papers devoted to rough sets exceeds half a thousand. However, only that of [10], is of special interest for us: the Alexandroff topologies and quasiorders appear there. Following [10], instead of equivalence relation θ one can apply any quasiorder ρ defined on a set of discourse A, so that given $U \subseteq A,\ U_\rho = \{a \in A : [a)_\rho \subseteq U\}$ and $U^\rho = \{a \in A : [a)_\rho \cap U \neq \emptyset\}$, where for any $a \in A,\ [a)_\rho = \{x \in A : a\rho x\} = (a]_{\rho^\sim}$. The pair (U_ρ, U^ρ) of such approximations for given $U \subseteq A$ is treated as a rough set. [10] proves that the poset of all quasiorder-based rough sets $(RS, \leq)$ is a complete sublattice of the lattice $(P(A), \subseteq) \times (P(A), \subseteq)$.

For the sake of clarity, let us denote by I_ρ the Alexandroff interior operation associated with a quasiorder ρ (this is a mapping from $ResInt(A)$ denoted by $f(\rho)$, cf. Corollary 2) and by C_ρ - the residual of I_ρ. So (I_ρ, C_ρ) is a residuated pair being at once an interior-closure pair of associated operations on the family DW_ρ of all the downward subsets of A with respect to ρ (the family DW_ρ, up till now denoted by $f_0(\rho)$, is the Alexandroff topology associated uniquely with ρ). So for any $U \subseteq A$ we have:

$I_\rho(U) = \{a \in A : (a]_\rho \subseteq U\} = \bigcup\{(a]_\rho : a \in A \ \&\ (a]_\rho \subseteq U\}$ = the greatest downward set with respect to ρ contained in U,

$C_\rho(U) = \bigcup\{(a]_\rho : a \in U\} = \{a \in A : [a)_\rho \cap U \neq \emptyset\} = \bigcup\{(a]_\rho : a \in A \ \&\ [a)_\rho \cap U \neq \emptyset\}$ = the least downward set with respect to ρ containing the set U.

Furthermore, the mappings $I_{\rho^\sim},\ C_\rho$ are the interior and closure operations of the Alexandroff topological space $(A, DW_{\rho^\sim})$ (whose topology is $DW_{\rho^\sim}$ - the family of all the upward sets with respect to ρ). Obviously, $I_\rho,\ C_{\rho^\sim}$ are the interior and closure operations of the dual Alexandroff space: (A, DW_ρ). In case of an equivalence relation θ on A we have $\theta^\sim = \theta$ so the interior-closure pair (I_θ, C_θ) of associated operations on the family DW_θ (forming a complete field of subsets of A) coincides with the pair of interior and closure operations of the same topological space (A, DW_θ) (which is well-known for logicians who proceed to an algebraic semantics from the Kripke one for the modal logic S5, cf. for example [3]).

It is easily seen that the Pawlak's approximations of $U \subseteq A$ may be written as: $U_\theta = I_\theta(U)$ and $U^\theta = C_\theta(U)$. Therefore, $RS = \{(I_\theta(U), C_\theta(U)) :$

$U \subseteq A\} \subseteq DW_\theta \times DW_\theta$. The above mentioned Pawlak's equalities can be rewritten in the form:

$I_\theta(-U) = -C_\theta(U)$,
$C_\theta(-U) = -I_\theta(U)$,
$I_\theta(I_\theta(U)) = C_\theta(I_\theta(U)) = I_\theta(U)$,
$C_\theta(C_\theta(U)) = I_\theta(C_\theta(U)) = C_\theta(U)$.

Now, in the case of generalization contained in [10], we can write for any quasiorder ρ on A and any $U \subseteq A:\ U_\rho = I_{\rho^\sim}(U)$ and $U^\rho = C_\rho(U)$. So the rough set of U is the pair composed out of the greatest open subset contained in U and the least closed one containing U with respect to the Alexandroff space $(A, DW_{\rho^\sim})$. Obviously, first two mentioned Pawlak's equalities are preserved for any quasiorder ρ put there instead of θ:

$I_{\rho^\sim}(-U) = -C_\rho(U)$,
$C_\rho(-U) = -I_{\rho^\sim}(U)$.

The last two (conjunctions of) equalities for a quasiorder other than an equivalence relation fail, i.e., the equalities: $C_\rho(I_{\rho^\sim}(U)) = I_{\rho^\sim}(U)$ and $I_{\rho^\sim}(C_\rho(U)) = C_\rho(U)$, do not hold for any $U \subseteq A$. Notice that the idea of Pawlak to approximate a vague set by two crisp sets built up from the same bulding blocks treated as atoms or units of measure, is not preserved in [10]. There the lower approximation of a vague set U is built up from the "blocks": $[a)_\rho,\ a \in A$, and the upper one is built up from diferent "blocks": $(a]_\rho,\ a \in A$, in the following sense: $U_\rho = I_{\rho^\sim}(U) = \bigcup\{[a)_\rho : a \in A \ \&\ [a)_\rho \subseteq U\}$ and $U^\rho = C_\rho(U) = \bigcup\{(a]_\rho : a \in U\}$. In the case of Pawlak approach the building blocks are of the form: $[a]_\theta : a \in A$, and $U_\theta = \bigcup\{[a]_\theta : a \in A \ \&\ [a]_\theta \subseteq U\}$, $U^\theta = \bigcup\{[a]_\theta : a \in U\}$. Why is the same form of bulding blocks so important? The very idea of Pawlak is to approximate a vague set U by taking - adopting the same measure - the nearest crisp sets to U from down and up. The same kind of blocks in lower and upper approximations guarantees the same measure. However, in the approach adopted in [10] the lower approximation of a set U is the nearest from down crisp set to U according to one measure, and the upper approximations of U is the nearest from up crisp set to U according to another measure. It seems that the idea contained in [10] is different from that one of Pawlak, however it is equally simple and nice: in order to approximate a vague set U one should take a reasonable topological space of which U is the subset and put the greatest open subset of U and the least closed superset of U as the approximations.

Our proposal goes along the lines of Pawlak's idea. We simply put $U_\rho = I_\rho(U)$ and $U^\rho = C_\rho(U)$ for any quasiorder ρ defined on A and any $U \subseteq A$. The sets $I_\rho(U),\ C_\rho(U)$ are the nearest ones to U from down and up with respect to inclusion, respectively, according to the same measure the units of which are the atoms: $(a]_\rho,\ a \in A$. In other words, as the approximations of U we take the downward subsets of A which are the nearest ones to U from down and up with respect to inclusion. Obviously, such approximations are the appropriate values of interior and closure operations associated on the family of all the downward subets of A. We are convinced that to keep to the spirit of Pawlak's approach, as the possible approximatizations, the values of interior-closure operations associated on a subset of $P(A)$ should be considered rather than the values of interior and closure operations in some topological space.

Acknowledgement. The author is grateful to both referees for valuable bibliographical hints.

References

[1] P. Alexandroff, *Diskrete Räume*, **Mat. Sbornik (Recueil Math.)** 2 (1937), pp. 501–517.

[2] F. G. Arenas, *Alexandroff spaces*, **Acta Math. Univ. Comenianae** 68(1) (1999), pp. 17–25.

[3] P. Blackburn P, M. de Rijke, Y. Venema, **Modal Logic**, Cambridge 2002.

[4] T. S. Blyth, **Lattices and Ordered Algebraic Structures**, Springer 2005.

[5] P. Cousot, R. Cousot, *A constructive characterization of the lattices of all retractions, preclosure, quasi-closure and closure operators on a complete lattice*, **Portugaliae Mathematica** 38(1979), pp. 185–198.

[6] K. Denecke, M. Erné, S. L. Wismath (eds.), **Galois Connections and Applications**, Kluwer 2004.

[7] F. Domenach, B. Leclerc, *Biclosed binary relations and Galois connections*, **Order** 18 (2001), pp. 89–104.

[8] M. Erné, J. Koslowski, A. Melton, G. E. Strecker, *A Primer on Galois Connections*, *Annals of the New York Academy of Sciences*, vol. 704 (1993), pp. 103–125.

[9] J. Järvinen, M. Kondo, J. Kortelainen, *Modal-Like Operators in Boolean Lattices, Galois Connections and Fixed Points*, **Fundamenta Informaticae** 76 (2007), pp. 129–145.

[10] J. Järvinen, S. Radeleczki, L. Veres, *Rough sets determined by quasiorders*, **Order** 26 (2009), pp. 337–355.

[11] M. C. McCord, *Singular homology and homotopy groups of finite topological spaces*, **Duke Math. Journal** 33(3) (1966), pp. 465–474.

[12] Z. Pawlak, *Rough sets*, **International Journal of Computer and Information Sciences** 11 (1982), pp. 341–356.

[13] Z. Pawlak, A. Skowron, *Rudiments of rough sets*, **Information Sciences** 177 (2007), pp. 3–27.

Department of Logic
University of Łódź
e-mail: marnowak@filozof.uni.lodz.pl