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CONGRUENCES AND IDEALS IN A DISTRIBUTIVE LATTICE WITH RESPECT TO A DERIVATION

Abstract

Two types of congruences are introduced in a distributive lattice, one in terms of ideals generated by derivations and the other in terms of images of derivations. An equivalent condition is derived for the corresponding quotient algebra to become a Boolean algebra. An equivalent condition is obtained for the existence of a derivation.

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Introduction

In the past several years, there has been an ongoing interest in derivations of rings [3, 7] as well as derivations of lattices [4, 9]. Given a lattice $(L, \vee, \wedge)$, we call derivation of L any self-mapping $d : L \longrightarrow L$ satisfying the following properties for all $x, y \in L$:

$$\begin{aligned} d(x \wedge y) &= d(x) \wedge y = x \wedge d(y) \\ d(x \vee y) &= d(x) \vee d(y) \end{aligned}$$

However, it can be formally stated for every algebraic structure endowed with two binary operations. In [8], these ideas have been introduced and main properties of derivations in lattices are established by G. Szasz. On the other hand, the study of congruence relations on lattices had became a special interest to many authors. In the paper [5], G. Gratzer and

E.T. Schmidt also studied an inter-relation between ideals and congruence relations in a lattice.

In this paper, two types of congruences, are introduced in a distributive lattice, both are defined in terms of derivations. The main aim of this paper is to obtain a necessary and sufficient condition for the quotient algebra L/θ (where θ is one of the congruences) to become a Boolean algebra. If L/θ is a Boolean algebra, then it is proved that θ is the largest congruence having a congruence class. Another congruence relation is introduced on a distributive lattice in terms of the images of derivations. Some useful properties of these congruence relations are then studied. For any ideal I, a necessary and sufficient condition is derived for the existence of a derivation d such that $d(L) = I$.

1. Preliminaries

In this section, we recall certain definitions and important results mostly from [1], [2], [4], [8]and [9], those will be required in the text of the paper.

Definition 1.1. [1] An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a lattice if for all $x, y, z \in L$, it satisfies the following properties

(1) $x \wedge x = x$, $x \vee x = x$
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$

Definition 1.2. [1] A lattice L is called distributive if for all $x, y, z \in L$ it satisfies the following properties

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

The least element of a distributive lattice is denoted by 0. Throughout this article L stands for a distributive lattice with 0, unless otherwise mentioned.

Definition 1.3. [2] Let $(L, \wedge, \vee)$ be a lattice. A partial ordering relation $\leq$ is defined on L by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Definition 1.4. [2] A non-empty subset I of L is called an ideal(filter) of L if $a \vee b \in A (a \wedge b \in A)$ and $a \wedge x \in A (a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$.

Definition 1.5. [2] A binary relation θ defined on L is a congruence on L if and only if it satisfies the following conditions:

1) θ is an equivalence relation on L
2) $(a, b) \in \theta$ implies $(a \wedge c, b \wedge c), (a \vee c, b \vee c) \in \theta$

The kernel of the congruence θ is defined as $Ker\ \theta = \{x \in L \mid (x, 0) \in \theta\}$

Definition 1.6. [4] Let $(L, \vee, \wedge)$ be a distributive lattice. A self-mapping $d : L \longrightarrow L$ is called a derivation of L if for all $x, y \in L$ it satisfies the following properties:

1) $d(x \wedge y) = d(x) \wedge y = x \wedge d(y)$
2) $d(x \vee y) = d(x) \vee d(y)$

The kernel of the derivation is the set $Ker\ d = \{x \in L \mid d(x) = 0\}$

Lemma 1.7. *[4] Let d be a derivation of L. Then for any $x, y \in L$, we have*

(1) $d(0) = 0$
(2) $d(x) \leq x$
(3) $d^2(x) = d(x)$
(4) $x \leq y \Rightarrow d(x) \leq d(y)$

2. The congruence θ_d

In this section, we introduce a congruence in terms of derivations of distributive lattices and obtain a necessary and sufficient condition for the quotient algebra of this congruence to become a Boolean algebra.

Definition 2.1. Let d be a derivation of L. For any $a \in L$, define the set $(a)^d$ as follows:

$$(a)^d = \{x \in L \mid a \wedge x \in Ker\ d\} = \{x \in L \mid a \wedge d(x) = 0\}$$

If $a \in Ker\ d$, then clearly $(a)^d = L$. Otherwise, Let $a \notin Ker\ d$. Suppose $a \in (a)^d$. Hence $d(a) = a \wedge d(a) = 0$, which is a contradiction. Hence $a \notin (a)^d$.

Some more basic properties can be observed in the following lemma.

LEMMA 2.2. *Let d be a derivation of L. Then for any $a, b \in L$, we have the following:*

(1) $(a)^d$ *is an ideal in* L
(2) $a \leq b$ *implies* $(b)^d \subseteq (a)^d$
(3) $(a \vee b)^d = (a)^d \cap (b)^d$

PROOF: (1). Clearly $0 \in (a)^d$. Let $x, y \in (a)^d$. Then we get $a \wedge d(x \vee y) = a \wedge (d(x) \vee d(y)) = (a \wedge d(x)) \vee (a \wedge d(y)) = 0 \vee 0 = 0$. Hence $x \vee y \in (a)^d$. Again, let $x \in (a)^d$ and $r \in L$. Then $a \wedge d(x \wedge r) = a \wedge d(x) \wedge r = 0 \wedge r = 0$. Hence $a \wedge r \in (a)^d$. Therefore $(a)^d$ is an ideal of L.

(2). Suppose $a \leq b$. Then we get $a = a \wedge b$. Let $x \in (b)^d$. Then we have $b \wedge d(x) = 0$. Now $a \wedge d(x) = a \wedge b \wedge d(x) = a \wedge 0 = 0$. Therefore $x \in (a)^d$.

(3). For any $a, b \in L$, we always have $(a \vee b)^d \subseteq (a)^d \cap (b)^d$. Conversely, let $x \in (a)^d \cap (b)^d$. Then we get $a \wedge d(x) = 0$ and $b \wedge d(x) = 0$. Now $(a \vee b) \wedge d(x) = (a \wedge d(x)) \vee (b \wedge d(x)) = 0 \vee 0 = 0$. Therefore $x \in (a \vee b)^d$. □

LEMMA 2.3. *Let d be a derivation of L. For any $a, b, c \in L$, we have*

(1) $(a)^d = (b)^d$ *implies* $(a \wedge c)^d = (b \wedge c)^d$
(2) $(a)^d = (b)^d$ *implies* $(a \vee c)^d = (b \vee c)^d$

PROOF: (1). Assume that $(a)^d = (b)^d$. Let $x \in (a \wedge c)^d$. Then $a \wedge c \wedge d(x) = 0$. Hence $a \wedge d(x \wedge c) = a \wedge d(x) \wedge c = 0$. Thus we get $x \wedge c \in (a)^d = (b)^d$. Therefore $b \wedge d(x) \wedge c = b \wedge d(x \wedge c) = 0$. Thus $b \wedge c \wedge d(x) = 0$. Therefore $x \in (b \wedge c)^d$. Thus $(a \wedge c)^d \subseteq (b \wedge c)^d$. Similarly, we can get $(b \wedge c)^d \subseteq (a \wedge c)^d$.

(2). Let $(a)^d = (b)^d$. By above Lemma, $(a \vee c)^d = (a)^d \cap (c)^d = (b \vee c)^d$. □

In the following, a binary relation is introduced on a distributive lattice with respect to a derivation.

DEFINITION 2.4. Let d be a derivation of L. Then for any $x, y \in L$, define a relation θ_d on L with respect to d, as $(x, y) \in \theta \Leftrightarrow (x)^d = (y)^d$

The following proposition is a direct consequence of the above Lemma.

PROPOSITION 2.5. *For any derivation d of L, the binary relation θ_d defined on L is a congruence relation on L.*

PROOF: Clearly θ_d is an equivalence relation. Let $x, y \in L$ such that $(x, y) \in \theta_d$. Then by above lemma, for any $c \in L$, we get $(x \wedge c, y \wedge c), (x \vee c, y \vee c) \in \theta_d$. □

By a congruence class $\theta(x)$(for any $x \in L$)of L with respect to θ, we mean the set $\theta(x) = \{t \in L \mid (x, t) \in \theta\}$. Let us denote the set of all congruence classes of L by L/θ. Then it can be easily observed that $(L/\theta, \wedge, \vee)$ is a distributive lattice in which the operations $\wedge$ and $\vee$ are given as follows:

$$\begin{aligned} \theta(x) \wedge \theta(y) &= \theta(x \wedge y) \\ \theta(x) \vee \theta(y) &= \theta(x \vee y) \qquad \text{for all } x, y \in L. \end{aligned}$$

The concept of kernel elements is now introduced in the following.

DEFINITION 2.6. An element $x \in L$ is called a kernel element if $(x)^d = Ker\ d$. Let us denote the set of all kernel elements of L by $\mathcal{K}_d$.

LEMMA 2.7. *For any derivation d of L, we have the following:*

(1) *$\mathcal{K}_d$ is a congruence class with respect to θ_d*
(2) *$Ker\ d \subseteq (x)^d$ for all $x \in L$*
(3) *$\mathcal{K}_d$ is closed under $\wedge$ and $\vee$ of L*
(4) *$\mathcal{K}_d$ is a filter of L, whenever $\mathcal{K}_d \neq \emptyset$*

PROOF: (1). It is clear.

(2). Let $a \in Ker\ d$. Then $d(a) = 0$ and hence $x \wedge d(a) = 0$ for all $x \in L$. Thus $a \in (x)^d$ for all $x \in L$. Therefore $Ker\ d \subseteq (x)^d$ for all $x \in L$.

(3). Let $a, b \in \mathcal{K}_d$. Then we get $(a)^d = (b)^d = Ker\ d$. Then $(a \vee b)^d = (a)^d \cap (b)^d = Ker\ d \cap Ker\ d = Ker\ d$. Hence $a \vee b \in \mathcal{K}_d$. Clearly $Ker\ d \subseteq (a \wedge b)^d$. Conversely, let $x \in (a \wedge b)^d$. Then

$$\begin{aligned} a \wedge b \wedge d(x) = 0 &\Rightarrow d(x) \wedge a \wedge b = 0 \\ &\Rightarrow d(x \wedge a) \wedge b = 0 \\ &\Rightarrow x \wedge a \in (b)^d = Ker\ d \end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \quad d(x \wedge a) = 0\\
&\Rightarrow \quad d(x) \wedge a = 0\\
&\Rightarrow \quad a \wedge d(x) = 0\\
&\Rightarrow \quad x \in (a)^d = Ker\ d
\end{aligned}
$$

Hence $(a \wedge b)^d \subseteq Ker\ d$. Thus $(a \wedge b)^d = Ker\ d$. Therefore $a \wedge b \in \mathcal{K}_d$.

(4). Let $a, b \in \mathcal{K}_d$. Then we have $(a)^d = Ker\ d$ and hence by (3), we get $a \wedge b \in \mathcal{K}_d$. For $x \in L$ and $a \in \mathcal{K}_d$, we get $(x \vee a)^d = (x)^d \cap (a)^d = (x)^d \cap Ker\ d = Ker\ d$. Hence $x \vee a \in \mathcal{K}_d$. Therefore $\mathcal{K}_d$ is a filter of L. □

In the following, a necessary and sufficient condition is derived for the quotient algebra L/θ_d to become a Boolean algebra.

THEOREM 2.8. *Let d be a derivation of L. Then L/θ_d is a Boolean algebra if and only if to each $x \in L$, there exists $y \in L$ such that $x \wedge y \in Ker\ d$ and $x \vee y \in \mathcal{K}_d$.*

PROOF: We first prove that $Ker\ d$ is the smallest congruence class and $\mathcal{K}_d$ is the largest congruence class in L/θ_d. Clearly $Ker\ d$ is a congruence class of L/θ_d. Since $Ker\ d$ is an ideal, we get that for any $a \in Ker\ d$ and $x \in L$, we have $a \wedge x \in Ker\ d$. Hence $\theta_d(a) \wedge \theta_d(x) = \theta_d(a \wedge x) = \theta_d(a) = Ker\ d$. This is true for all $x \in L$. Therefore $\theta_d(a) = Ker\ d$ is the smallest congruence class of L/θ_d. Again, clearly $\mathcal{K}_d$ is a congruence class of L/θ_d. Let $a \in \mathcal{K}_d$ and $x \in L$. Since $\mathcal{K}_d$ is a filter, we get that $x \vee a \in \mathcal{K}_d$. Therefore $(x \vee a)^d = Ker\ d$. We now prove that $\mathcal{K}_d$ is the greatest congruence class of L/θ_d. For any $a \in \mathcal{K}_d$ and $x \in L$, we get that $\theta_d(x) \vee \theta_d(a) = \theta_d(x \vee a) = \theta_d(a)$. Therefore $\mathcal{K}_d$ is the greatest congruence class of L/θ_d. We now prove the main part of the Theorem.

Assume that L/θ_d is a Boolean algebra. Let $x \in L$ so that $\theta_d(x) \in L/\theta_d$. Since L/θ_d is a Boolean algebra, there exists $\theta_d(y) \in L/\theta_d$ such that $\theta_d(x \wedge y) = \theta_d(x) \cap \theta_d(y) = Ker\ d$ and $\theta_d(x \vee y) = \theta_d(x) \vee \theta_d(y) = \mathcal{K}_d$. Hence $x \wedge y \in Ker\ d$ and $x \vee y \in \mathcal{K}_d$. Converse can be proved in a similar way. □

We conclude this section with the derivation of a sufficient condition for the congruence θ_d to become the greatest congruence with congruence class $\mathcal{K}_d$.

Theorem 2.9. *Let d be a derivation of L. If L/θ_d is a Boolean algebra, then θ_d is the largest congruence relation having congruence class $\mathcal{K}_d$.*

Proof: Clearly θ_d is a congruence with $\mathcal{K}_d$ as a congruence class. Let θ be any congruence with $\mathcal{K}_d$ as a congruence class. Let $(x, y) \in \theta$. Then for any $a \in L$, we can have

$$\begin{aligned}(x, y) \in \theta \quad &\Rightarrow \quad (x \vee a, y \vee a) \in \theta \\ &\Rightarrow \quad x \vee a \in \mathcal{K}_d \ iff \ y \vee a \in \mathcal{K}_d \\ &\Rightarrow \quad (x \vee a)^d = Ker\ d\ iff (y \vee a)^d = Ker\ d \\ &\Rightarrow \quad (x)^d \cap (a)^d = Ker\ d\ iff\ (y)^d \cap (a)^d = Ker\ d \qquad --(\star)\end{aligned}$$

Since L/θ_d is a Boolean algebra, by above Theorem, there exists $x', a' \in L$ such that $x \wedge x', a \wedge a' \in Ker\ d$ and $(x \vee x')^d = Ker\ d, (a \vee a')^d = Ker\ d$. Hence $x' \in (x)^d$ and $a' \in (a)^d$ which implies that $x' \wedge a' \in (x)^d \cap (a)^d = Ker\ d$. Therefore $a' \in (x')^d$. Similarly, we can get $a' \in (y')^d$ for a suitable $y' \in L$. Thus by above condition $(\star)$, we get

$$\begin{aligned}a' \in (x')^d \ \text{ iff } \ a' \in (y')^d \quad &\Rightarrow \quad (x')^d = (y')^d \\ &\Rightarrow \quad (x', y') \in \theta_d \\ &\Rightarrow \quad x' \in \mathcal{K}_d\ iff\ y' \in \mathcal{K}_d \\ &\Rightarrow \quad (x')^d = Ker\ d\ iff\ (y')^d = Ker\ d \\ &\Rightarrow \quad (x \vee x')^d = (x)^d\ iff\ (y \vee y')^d = (y)^d \\ &\Rightarrow \quad (x)^d = Ker\ d\ iff\ (y)^d = Ker\ d \\ &\Rightarrow \quad (x)^d = (y)^d \\ &\Rightarrow \quad (x, y) \in \theta_d\end{aligned}$$

□

3. The congruence θ^d

In this section, a special type of congruence is introduced in terms of a derivation. Some properties of these congruences are studied. A necessary and sufficient condition is obtained for the existence of a derivation.

Definition 3.1. Let d be a derivation of L. Then define a relation θ^d with respect to d on L by $(x, y) \in \theta^d$ if and only if $d(x) = d(y)$ for all $x, y \in L$.

In the following sequence of Lemmas, some preliminary properties of the binary relation θ^d are observed.

Lemma 3.2. *For any derivation d of L, we have the following:*

(1) θ^d *is a congruence relation on* L
(2) $Ker\ \theta^d = Ker\ d$

Proof: (1). Clearly θ^d is an equivalence relation on L. Now let $(x, y) \in \theta^d$. Then we have $d(x) = d(y)$. Let c be an arbitrary element of L. Then we get $d(x \wedge c) = d(x) \wedge c = d(y) \wedge c = d(y \wedge c)$. Hence $(x \wedge c, y \wedge c) \in \theta^d$. Again $d(x \vee c) = d(x) \vee d(c) = d(y) \vee d(c) = d(y \vee c)$. Hence $(x \vee c, y \vee c) \in \theta^d$. Therefore θ^d is a congruence relation on L.

(2). For any derivation d of L, we have $Ker\ \theta^d = \{x \in L \mid (x, 0) \in \theta^d\} = \{x \in L \mid d(x) = d(0) = 0\} = Ker\ d$. □

Lemma 3.3. *Let d be a derivation of L. Then we have the following:*

(1) $d(x) = x$ *for all* $x \in d(L)$
(2) If $(x, y) \in \theta^d$ *and* $x, y \in d(L)$, *then* $x = y$

Proof: (1). Let $x \in d(L)$. Then we have that $x = d(a)$ for some $a \in L$. Since d is a derivation, we get that $d(x) = d^2(x) = d(d(x)) = d(a) = x$.

(2). Let $(x, y) \in \theta^d$. Then we have $d(x) = d(y)$. Since d is a derivation and $x, y \in d(L)$, we can get $x = y$. □

Finally, for a given ideal I, a necessary and sufficient condition is obtained for the existence of a derivation such that $d(L) = I$

Theorem 3.4. *Let I be an ideal of L. Then there exists a derivation d on L such that $d(L) = I$ if and only if there exists a congruence relation θ on L such that $I \cap [x]_\theta$ is a singleton set for all $x \in L$.*

Proof: Let d be a derivation of L such that $d(L) = I$. Then clearly θ^d is a congruence relation on L. Let $x \in L$. Since $d(x) = d^2(x)$, we get that $(x, d(x)) \in \theta^d$. Hence $d(x) \in [x]_{\theta^d}$. Also $d(x) \in d(L) = I$. Hence $d(x) \in I \cap [x]_{\theta^d}$. Therefore $I \cap [x]_{\theta^d} \neq \emptyset$. Suppose a, b be two elements in $I \cap [x]_{\theta^d}$. Then by above Lemma, $a = b$. Therefore $I \cap [x]_{\theta^d}$ is a singleton set.

Conversely, assume that there exists a congruence θ on L such that $I \cap [x]_\theta$ is a singleton set for every $x \in L$. Let x_0 be the single element of $I \cap [x]_\theta$. Define a map $d : L \longrightarrow L$ by $d(x) = x_0$ for all $x \in L$. Let $a, b \in L$. Then clearly $d(a \vee b) = x_0 = x_0 \vee x_0 = d(a) \vee d(b)$. Now $d(a \wedge b) = x_0 \in I \cap [a \wedge b]_\theta$. By the definition of d, we can get that $d(b) \in I$ and $d(d(b)) = d(b)$. Now we can obtain the following consequence:

$$\begin{aligned} d(d(b)) = d(b) &\Rightarrow (d(b), b) \in \theta \\ &\Rightarrow (a \wedge d(b), a \wedge b) \in \theta \\ &\Rightarrow a \wedge d(b) \in [a \wedge b]_\theta \\ &\Rightarrow a \wedge d(b) \in I \cap [a \wedge b]_\theta \qquad (\text{ since } d(b) \in I\ ) \end{aligned}$$

Since $d(a \wedge b) \in I \cap [a \wedge b]_\theta$ and $I \cap [a \wedge b]_\theta$ has a single element, we get $d(a \wedge b) = a \wedge d(b)$. Similarly, we can get that $d(a \wedge b) = d(a) \wedge b$. Thus $d(a \wedge b) = d(a) \wedge b = a \wedge d(b)$. Hence d is a derivation of L. □

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