Bulletin of the **S**ection of **L**ogic
Volume 42:1/2 (2013), pp. 69–81

Lei-Bo Wang

CLOSURE EXTENDED DOUBLE STONE ALGEBRAS

Abstract

The variety **CDS** of closure extended double Stone algebras consists of the algebras $(L; \wedge, \vee, ^*, ^+, ^\circ, 0, 1)$ of type $(2, 2, 1, 1, 1, 0, 0)$ where $(L; \wedge, \vee, ^*, ^+, 0, 1)$ is a double Stone algebra, $^\circ$ is a lattice endomorphism on L with $x \leq x^\circ = x^{\circ\circ}$ and the operations $x \mapsto x^*$, $x \mapsto x^+$ and $x \mapsto x^\circ$ are linked by the identities $x^{*\circ} = x^{\circ *}$ and $x^{+\circ} = x^{\circ +}$. In this paper, we characterize congruences on a **CDS**-algebra, and show that there are precisely seven non-isomorphic subdirectly irreducible members in the class of these algebras and give a complete description of them.

Keywords: double Stone algebra, congruence, subdirectly irreducible

1991 Mathematics Subject Classification 06D15

1. Introduction

A *(distributive) p-algebra* (or *lattice with pseudocomplementation*) is a (distributive) lattice L with a smallest element 0 together with a mapping $^* : L \to L$ such that $x \wedge y = 0 \iff y \leq x^*$. A *dual (distributive) p-algebra* is a (distributive) lattice L with a biggest element 1 together with a mapping $^+ : L \to L$ such that $x \vee y = 1 \iff y \geq x^+$. A *(distributive) double p-algebra* $(L; \wedge, \vee, ^*, ^+, 0, 1)$ (or *lattice with double pseudocomplementation*) is a lattice L such that $(L; \wedge, \vee, ^*, 0, 1)$ is (distributive) p-algebra and $(L; \wedge, \vee, ^+, 0, 1)$ is a dual (distributive) p-algebra. A special subclass of class of distributive double p-algebras $(L; ^*, ^+)$ is the class of *double Stone* *algebras* in which the unary operations * and $^+$ are satisfied the double Stone identities:

$$x^* \vee x^{**} = 1 \text{ and } x^+ \wedge x^{++} = 0.$$

For a double Stone algebra $(L;^*,^+)$, in what follows we write $L^* = \{x^* \mid x \in L\}$ and $L^+ = \{x^+ \mid x \in L\}$. The following rules of computation in a double Stone algebra $(L;^*,^+)$ will be needed and can easily be proved:

(1) $(\forall x \in L)\ x^* \leq x^+$;

(2) $(\forall x \in L)\ x^{+*} = x^{++} \leq x \leq x^{**} = x^{*+}$;

(3) $(\forall x, y \in L)\ (x \wedge y)^* = x^* \vee y^*$ and $(x \vee y)^+ = x^+ \wedge y^+$;

(4) $(\forall x \in L)\ x^* \vee x^{**} = 1$ and $x^+ \wedge x^{++} = 0$;

(5) $L^* = L^+$.

For the basic properties of distributive double p-algebras and double Stone algebras we refer the reader to [3] and [4].

In the previous paper [5], T. S. Blyth and Jie Fang introduced the notion of the extended Ockham algebras. Precisely, an *extended Ockham algebra* $L \equiv (L; \wedge, \vee, f,^\circ, 0, 1)$ is a bounded distributive lattice L with two unary operations f and $^\circ$ such that

(1) f is a dual lattice endomorphism with $f(1) = 0$ and $f(0) = 1$;
(2) $^\circ$ is a lattice endomorphism with $1^\circ = 1$ and $0^\circ = 0$;
(3) f and $^\circ$ commute.

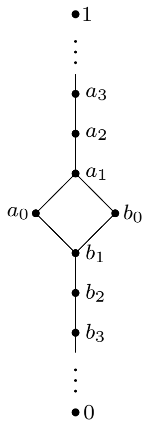
Here we shall consider another particular class of algebras that contains double Stone algebras. This subvariety is defined as follows.

Definition. By an *extended double Stone algebra* $L \equiv (L; \wedge, \vee,^*,^+,^\circ, 0, 1)$ we mean a bounded distributive lattice L together with three unary operations *, $^+$ and $^\circ$ such that:

(1) $(L;^*,^+)$ is a double Stone algebra;
(2) $^\circ$ is a lattice endomorphism with $1^\circ = 1$, $0^\circ = 0$;
(3) $x^{\circ *} = x^{*\circ}$ and $x^{\circ +} = x^{+\circ}$.

An extended double Stone algebra $L \equiv (L; \wedge, \vee,^*,^+,^\circ)$ in which $x \leq x^\circ = x^{\circ\circ}$ $(\forall x \in L)$ is said to be *closure*. We shall denote by **CDS** the variety of closure extended double Stone algebras.

Example 1. Consider the algebra $(L;^*,^+,^\circ)$ depicted as follows:



Let $^\circ : L \to L$ be described by

$0^\circ = 0,\ \ 1^\circ = 1,$
$(\forall i)\ a_i^\circ = a_{i+1},\ b_i^\circ = b_{i+1};$

let $^* : L \to L$ be described by

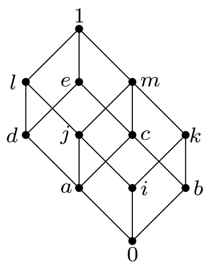
$0^* = 1,\ \ 1^* = 0,$
$(\forall i)\ a_i^* = b_i^* = 0;$

and let $^+ : L \to L$ be described by

$0^+ = 1,\ \ 1^+ = 0,$
$(\forall i)\ a_i^+ = b_i^+ = 1.$

Clearly, $(L;^*,^+)$ is a double Stone algebra and $^\circ$ is lattice endomorphism. We see that for $x \in L$, $x^{*\circ} = x^{\circ *}$, $x^{+\circ} = x^{\circ +}$. Thus $(L;^*,^+,^\circ)$ is an extended double Stone algebra.

Example 2. Consider the algebra $(L;^*,^+,^\circ)$ given as follows:



	0	a	b	c	d	e	i	j	k	l	m	1
$*$	1	k	l	i	k	i	e	b	d	b	0	0
$+$	1	1	l	l	k	i	e	e	d	b	d	0
$\circ$	0	d	b	e	d	e	i	l	k	l	1	1

By a simple calculation we can see that $(L;^*,^+,^\circ)$ is a **CDS**-algebra.

2. Congruences

For a **CDS**-algebra $(L;^*,^+,^\circ)$, in what follows we write $L^\circ = \{x^\circ \mid x \in L\} = \{x = x^\circ, x \in L\}$. Clearly, L° is a subalgebra of L. We begin with the following basic result.

THEOREM 1. *If* $(L;^*,^+,^\circ) \in \mathbf{CDS}$ *then we have the following properties:*

(1) $(\forall x \in L)$ $x^* = x^{*\circ} = x^{\circ *}$, $x^+ = x^{+\circ} = x^{\circ +}$ *and* $L^* = L^+ \subseteq L^\circ$;

(2) $(\forall x \in L)$ $x^{++} \leq x \leq x^\circ \leq x^{**}$;

(3) $(\forall x \in L)$ $x^* \neq x^\circ$ *and* $x^+ \neq x^\circ$;

(4) $(\forall x \in L)$ $x^\circ = 1 \Rightarrow x^* = x^+ = 0$.

PROOF: (1) For any $x \in L$, since $x \leq x^\circ$ gives $x^* \geq x^{*\circ}$. On the other hand, $x^* \leq x^{*\circ}$ and so $x^* = x^{*\circ} = x^{\circ *}$. Hence, we obtain $L^* \subseteq L^\circ$. Similarly, we can obtain $x^+ = x^{+\circ} = x^{\circ +}$ for any $x \in L$.

(2) Since $x \leq x^{**}$, $x^\circ \leq x^{\circ **} = x^{**}$ by (1) for every $x \in L$. Hence, $x^{++} \leq x \leq x^\circ \leq x^{**}$.

(3) Suppose that $x^* = x^\circ$ for any $x \in L$. By (1) and (2), it follows the contradiction $x^* = x^{**}$. In a similar way, we must have $x^+ \neq x^\circ$.

(4) It is obvious from (1). □

We shall now be concerned with the congruences on a **CDS**-algebra.

By a *congruence* on a **CDS**-algebra $(L;^*,^+,^\circ)$ we shall mean a lattice congruence θ such that

$$(x, y) \in \theta \Rightarrow (x^*, y^*) \in \theta,\ \ (x^+, y^+) \in \theta \ \text{ and } \ (x^\circ, y^\circ) \in \theta.$$

Through what follows, for a **CDS**-algebra $(L;^*,^+,^\circ)$ we shall denote by $\mathrm{Con}_{lat}\, L$ the lattice of lattice congruences of L; and by $\mathrm{Con}\, L$ the lattice of congruences of $(L;^*,^+,^\circ)$. The equality relation and the universal relation on L are always denoted by ω and ι, respectively.

If $(L;^*,^+)$ is a double Stone algebra, then the equivalence relations G^*, G^+ and G on L given by

$$(x, y) \in G^* \iff x^* = y^*,$$
$$(x, y) \in G^+ \iff x^+ = y^+$$

and

$$(x, y) \in G \iff x^* = y^*, x^+ = y^+$$

are determination congruences on L.

In a **CDS**-algebra $(L;^*,^+,^\circ)$, we define an equivalence relation Φ on L given by

$$(x, y) \in \Phi \iff x^\circ = y^\circ.$$

Clearly, Φ is a congruence on L, $\Phi \leq G^*$, $\Phi \leq G^+$ and $\Phi \leq G^* \wedge G^+ = G$.

The following result is an analogous form of [6, Theorem 2]

THEOREM 2. *If* $(L;^*,^+,^\circ) \in \mathbf{CDS}$ *and* $\alpha, \beta \in \operatorname{Con} L$ *then*

$$\alpha|_{L^\circ} = \beta|_{L^\circ} \iff \alpha \vee \Phi = \beta \vee \Phi.$$

PROOF: $\Rightarrow$: Suppose that $\alpha|_{L^\circ} = \beta|_{L^\circ}$ and $(x, y) \in \alpha \vee \Phi$. Then there exist elements $x = x_0, x_1, ..., x_n = y$ of L such that

$$x = x_0 \equiv x_1 \equiv ... \equiv x_n = y$$

where $(x_i, x_{i+1}) \in \alpha$ or $(x_i, x_{i+1}) \in \Phi$. The latter gives then $x_i^\circ = x_{i+1}^\circ$. Hence it follows that $(x^\circ, y^\circ) \in \alpha$. Obviously, we have $(x^\circ, y^\circ) \in \alpha|_{L^\circ} = \beta|_{L^\circ}$, it follows that $(x, y) \in \beta \vee \Phi$ and so $\alpha \vee \Phi \leq \beta \vee \Phi$. The reverse inclusion is established similarly.

$\Leftarrow$: Suppose that $\alpha \vee \Phi = \beta \vee \Phi$ and let $x, y \in L^\circ$ be such that $(x, y) \in \alpha$. Hence $(x, y) \in \alpha \vee \Phi = \beta \vee \Phi$. Then there exist elements $x = x_0, x_1, ..., x_n = y$ of L such that

$$x = x_0 \equiv x_1 \equiv ... \equiv x_n = y$$

where $(x_i, x_{i+1}) \in \beta$ or $(x_i, x_{i+1}) \in \Phi$. The latter gives then $x_i^\circ = x_{i+1}^\circ$. Thus $(x^\circ, y^\circ) \in \beta$. Since $x, y \in L^\circ$ we have $x = x^\circ$ and $y = y^\circ$. Hence $(x, y) \in \beta$ and so $\alpha|_{L^\circ} \leq \beta|_{L^\circ}$. The reverse inclusion is established similarly. □

Let $(L;^*,^+,^\circ) \in \mathbf{CDS}$ and $a, b \in L$ with $a \leq b$. We shall denote by $\theta(a, b)$ the principal congruence generated by $\{a, b\}$, i.e. the smallest congruence on $(L;^*,^+,^\circ)$ that identifies a and b. The corresponding principal lattice congruence will be denoted by $\theta_{lat}(a, b)$. We now give a description on the principal congruences of a **CDS**-algebra as follows.

THEOREM 3. *Let* $(L;^*,^+,^\circ) \in \mathbf{CDS}$ *and* $a, b \in L$ *with* $a \leq b$. *Then*

$$(\star) \qquad \theta(a, b) = \theta_{lat}(a, b) \vee \theta_{lat}(a^\circ, b^\circ) \vee \theta_{lat}(b^*, a^*) \vee \theta_{lat}(b^+, a^+).$$

PROOF: Let $\varphi(a, b)$ denote the right side of the equality $(\star)$. Observe that $(a, b) \in \theta(a, b)$ gives $(a^\circ, b^\circ) \in \theta(a, b)$, $(a^*, b^*) \in \theta(a, b)$ and $(a^+, b^+) \in \theta(a, b)$. Thus we have $\varphi(a, b) \leq \theta(a, b)$. To see the reverse inequality, it suffices to show that $\varphi(a, b)$ preserve the unary operations *, $^+$ and $^\circ$. For doing so, we need only observe the following facts.

(1) If $(x,y)\in\theta_{lat}(a,b)$, then $(x^\circ,y^\circ)\in\theta_{lat}(a^\circ,b^\circ)$, $(x^*,y^*)\in\theta_{lat}(b^*,a^*)$ and $(x^+,y^+)\in\theta_{lat}(b^+,a^+)$.

Suppose that $(x,y)\in\theta_{lat}(a,b)$. Then $x\wedge a=y\wedge a$ and $x\vee b=y\vee b$. Thus we have $x^\circ\wedge a^\circ=y^\circ\wedge a^\circ$ and $x^\circ\vee b^\circ=y^\circ\vee b^\circ$, and so $(x^\circ,y^\circ)\in\theta_{lat}(a^\circ,b^\circ)$. Similarly, we have $(x^*,y^*)\in\theta_{lat}(b^*,a^*)$ and $(x^+,y^+)\in\theta_{lat}(b^+,a^+)$.

By a similar argument as that of (1), and noting that $(L^*;^*)$ is boolean, $x^*=x^{*\circ}=x^{\circ *}$ and $x^+=x^{+\circ}=x^{\circ +}$, we can have the following:

(2) If $(x,y)\in\theta_{lat}(a^\circ,b^\circ)$, then $(x^\circ,y^\circ)\in\theta_{lat}(a^\circ,b^\circ)$, $(x^*,y^*)\in\theta_{lat}(b^*,a^*)$ and $(x^+,y^+)\in\theta_{lat}(b^+,a^+)$;

(3) If $(x,y)\in\theta_{lat}(b^*,a^*)$, then (x°,y°), (x^*,y^*) and $(x^+,y^+)\in\theta_{lat}(b^*,a^*)$;

(4) If $(x,y)\in\theta_{lat}(b^+,a^+)$, then (x°,y°), (x^*,y^*) and $(x^+,y^+)\in\theta_{lat}(b^+,a^+)$.

Then we have $\varphi(a,b)$ is a congruence on L.

We therefore have from the observations above that $\varphi(a,b)=\theta(a,b)$. □

By Theorem 3, the following corollary is immediate.

Corollary 1. *If $(L;^*,^+,^\circ)\in$ **CDS** and $a,b\in L$ with $a\le b$. Then we have the following properties:*

(1) *If $(a,b)\in G^*$, then $\theta(a,b)=\theta_{lat}(a,b)\vee\theta_{lat}(a^\circ,b^\circ)\vee\theta_{lat}(b^+,a^+)$;*
(2) *If $(a,b)\in G^+$, then $\theta(a,b)=\theta_{lat}(a,b)\vee\theta_{lat}(a^\circ,b^\circ)\vee\theta_{lat}(b^*,a^*)$;*
(3) *If $(a,b)\in G$, then $\theta(a,b)=\theta_{lat}(a,b)\vee\theta_{lat}(a^\circ,b^\circ)$;*
(4) *If $(a,b)\in\Phi$, then $\theta(a,b)=\theta_{lat}(a,b)$.*

Theorem 4. *Let $(L;^*,^+,^\circ)\in$ **CDS** and $a,b\in L$ with $a\le b$. Then*

$$\theta(a,b)=\theta_{lat}(a,b)\vee\theta_{lat}((a^\circ\vee b^*)\wedge b^+,(b^\circ\vee a^*)\wedge a^+).$$

Proof: Let $\varphi=\theta_{lat}(a,b)\vee\theta_{lat}((a^\circ\vee b^*)\wedge b^+,(b^\circ\vee a^*)\wedge a^+)$. It is obvious that $\varphi\le\theta(a,b)$ by Theorem 3. Observe that $(a^\circ\vee b^*)\wedge b^+\overset{\varphi}{\equiv}(b^\circ\vee a^*)\wedge a^+$ implies that $a^\circ\vee b^*\overset{\varphi}{\equiv}(a^\circ\vee b^*)\vee[(b^\circ\vee a^*)\wedge a^+]=(a^\circ\vee b^{\circ *})\vee[(b^\circ\vee a^{\circ *})\wedge a^{\circ +}]=b^\circ\vee a^*$ by Theorem 1(1), whence $b^\circ\overset{\varphi}{\equiv}b^\circ\wedge(a^\circ\vee b^*)=a^\circ$ and $a^*\overset{\varphi}{\equiv}a^*\wedge(a^\circ\vee b^*)=b^*$. Also, we have $b^+\overset{\varphi}{\equiv}b^+\vee[(b^\circ\vee a^*)\wedge a^+]=a^+$. Thus we have $\theta(a,b)\leqslant\varphi$. Consequently, $\theta(a,b)=\varphi$. □

3. Subdirectly irreducible algebras

We shall now consider the subdirectly irreducible algebras in **CDS**. An algebra L is *subdirectly irreducible*, if $\mathrm{Con}\, L \setminus \{\omega\}$ has a smallest element, called the *monolith* of $\mathrm{Con}\, L$. Dually, a congruence relation is called the *comonolith* if it is the largest element of $\mathrm{Con}\, L \setminus \{\iota\}$. In the special case when $\mathrm{Con}\, L = \{\omega, \iota\}$, the algebra L is said to be *simple*.

THEOREM 5. *If* $(L;{}^*, {}^+, {}^\circ) \in$ **CDS** *is subdirectly irreducible, then* $L^* = L^+ = \{0, 1\}$.

PROOF: Suppose that $|L^*| \geq 3$. Then there exists $a^* \in L^*$ with $0 < a^* < 1$. Thus it follows by Theorem 3 the contradiction that

$$\theta(0, a^*) \wedge \theta(a^*, 1) = \theta_{lat}(0, a^*) \wedge \theta_{lat}(a^*, 1) = \omega. \qquad \square$$

COROLLARY 2. *Let* $(L;{}^*, {}^+, {}^\circ)$ *be a subdirectly irreducible* **CDS**-*algebra, then we have the following properties:*

(1) $(\forall x \in L)\ x \neq 0 \Rightarrow x^* = 0$;

(2) $(\forall x \in L)\ x \neq 1 \Rightarrow x^+ = 1$.

PROOF: (1) Suppose that $x \neq 0$ for any $x \in L$, since $x \wedge x^* = 0$, by Theorem 5 we must have $x^* = 0$.

(2) The argument is similar to (1). $\square$

COROLLARY 3. *If* $(L;{}^*, {}^+, {}^\circ)$ *is a subdirectly irreducible* **CDS**-*algebra, then* L *has at most one atom (coatom).*

PROOF: If $a, b \in L$ with $a \neq b$ are atoms of L then $a > 0$ and $b > 0$ with $a \wedge b = 0$. This is impossible; for otherwise, it follows from the Corollary 2(1) the contradiction that $1 = 0^* = (a \wedge b)^* = a^* \vee b^* = 0 \vee 0 = 0$. Thus L has at most one atom.

In a similar way, we can show that L has at most one coatom. $\square$

THEOREM 6. *If* $(L;{}^*, {}^+, {}^\circ) \in$ **CDS** *is subdirectly irreducible, then we have the following statements:*

(1) *Every* Φ-*class contains at most two elements;*

(2) *Every* G-*class contains at most two elements of* L°.

Proof: (1) If there exist $a, b, c \in L$ with $a < b < c$ such that $a, b, c \in [x]\Phi$ for some $x \in L$, then by Corollary 1, $\theta(a,b) \wedge \theta(b,c) = \theta_{lat}(a,b) \wedge \theta_{lat}(b,c) = \omega$. This contradiction shows that (1) holds.

(2) The argument is similar to that of (1). □

The following result is similar to [2, Theorem 3.16].

Theorem 7. *If* $(L;^*, ^+, ^\circ) \in \mathbf{CDS}$ *is subdirectly irreducible and* $\Phi \neq \omega$, *then* Φ *is the monolith of* $\operatorname{Con} L$.

Proof: Let α be the monolith of $\operatorname{Con} L$. Then $\omega \prec \alpha \leq \Phi$, and so there exist $a, b \in L$ with $a \prec b$ such that $\theta(a,b) = \alpha$. Thus by Corollary 1, we have $\alpha = \theta(a,b) = \theta_{lat}(a,b)$. Since α is a principal lattice congruence it has a complement β in $\operatorname{Con}_{lat} L$. Then the lattice congruence $\beta \wedge \Phi \in \operatorname{Con} L$, and so $\beta \wedge \Phi = \omega$ or $\beta \wedge \Phi \geq \alpha$. The latter gives the contradiction that $\alpha = \alpha \wedge \beta \wedge \Phi = \alpha \wedge \beta = \omega$. Thus we must have $\beta \wedge \Phi = \omega$. But $\alpha \vee \beta = \iota$ and $\alpha \leq \Phi$ give $\beta \vee \Phi = \iota$. Hence Φ is the complement of β in $\operatorname{Con}_{lat} L$ and whence $\alpha = \Phi$. Thus Φ is the monolith of $\operatorname{Con} L$. □

Theorem 8. *If* $(L;^*, ^+, ^\circ) \in \mathbf{CDS}$ *is subdirectly irreducible, then* $|L^\circ| \leq 4$.

Proof: Suppose that L is subdirectly irreducible and let $|L^\circ| \geqslant 5$. Then L° must contain either an 5-element chain or two non-comparable elements. If, on the one hand, there exists an 5-element chain in L°, say $0 < a < b < c < 1$ with $a, b, c \in L^\circ$, whence it follows by Theorem 3 the contradiction that

$$\theta(a,b) \wedge \theta(b,c) = \theta_{lat}(a,b) \wedge \theta_{lat}(b,c) = \omega.$$

If, on the other hand, there exist $a, b \in L^\circ$ such that a and b are non-comparable, then by Corollary 3 we must have $a \wedge b \neq 0$ and $a \vee b \neq 1$. It follows from Corollary 1 and Corollary 2 the contradiction that

$$\theta(a \wedge b, a) \wedge \theta(a \wedge b, b) = \theta_{lat}(a \wedge b, a) \wedge \theta_{lat}(a \wedge b, b) = \omega.$$

It therefore follows from the above observations that $|L^\circ| \leqslant 4$. □

Corollary 4. *Let* $(L;^*, ^+, ^\circ) \in \mathbf{CDS}$ *be subdirectly irreducible. Then* L° *is subdirectly irreducible.*

Proof: Since $|L^\circ| \leqslant 4$ by Theorem 8, we observe the following three conditions:

(1) $L^\circ = \{0,1\}$. Clearly, $\operatorname{Con} L^\circ = \{\omega, \iota\}$ and so L° is simple.

(2) $L^\circ = \{0,a,1\}$ with $0 < a < 1$. In this case, by Corollary 2, we have $\operatorname{Con} L^\circ = \{\omega, \iota\}$ and so L° is simple.

(3) $L^\circ = \{0,a,b,1\}$ with $0 < a < b < 1$. In this case, we obtain by Corollary 2 again that $\operatorname{Con} L^\circ$ is a chain: $\omega \prec \theta(a,b) = \theta_{lat}(a,b) \prec \iota$.

Hence we obtain that L° is subdirectly irreducible. □

Corollary 5. *If* $(L;^*,^+,^\circ) \in \mathbf{CDS}$ *is subdirectly irreducible then* $|L| \leq 6$.

Proof: Since, by the Theorem 6(1) and Theorem 8, $L = [0]\Phi \cup [1]\Phi \cup [a]\Phi \cup [b]\Phi$ for some $a,b \in L$ and noting that $[0]\Phi = \{0\}$ and $[1]\Phi = \{1\}$, it then follows $|L| \leqslant 6$. □

Corollary 6. *If* $(L;^*,^+,^\circ) \in \mathbf{CDS}$ *is subdirectly irreducible then* $|[x]G| \leq 4$ *for some* $x \in L \backslash \{0,1\}$.

Proof: Since, by the Theorem 5 and Corollary 2, $L = [x]G \cup \{0,1\}$ for some $x \in L \backslash \{0,1\}$. It then follows $|[x]G| \leq 4$ by Corollary 5. □

Theorem 9. *Let* $(L;^*,^+,^\circ) \in \mathbf{CDS}$ *be subdirectly irreducible. Then* G *is the comonolith of* $\operatorname{Con} L$.

Proof: Suppose now that $\varphi \in \operatorname{Con} L$ is such that $\varphi \not\leq G$. Then there exist $a,b \in L$ with $(a,b) \in \varphi$ but $(a,b) \notin G$. Thus $(a^*,b^*) \in \varphi$ and $(a^+,b^+) \in \varphi$. By Theorem 5, we have $\{a^*,b^*\} \subseteq \{0,1\}$ and $\{a^+,b^+\} \subseteq \{0,1\}$. Since $(a,b) \notin G$, we must have $a^* \neq b^*$ or $a^+ \neq b^+$. Thus $\{a^*,b^*\} = \{0,1\}$ or $\{a^+,b^+\} = \{0,1\}$. It follows that $(0,1) \in \varphi$ whence $\varphi = \iota$. Consequently, G is the comonolith of $\operatorname{Con} L$. □

We now consider the structure of the lattice $\operatorname{Con} L$ of a subdirectly irreducible **CDS**-algebra.

Theorem 10. *Let* $(L;^*,^+,^\circ) \in \mathbf{CDS}$ *be subdirectly irreducible. Then*

(1) *If* $G^* = \omega$ *or* $G^+ = \omega$ *then* $\operatorname{Con} L = \{\omega, \iota\}$, *namely,* L *is simple;*

(2) *If* $\Phi = \omega$ *and* $G \neq \omega$ *then* $\operatorname{Con} L$ *is the chain:* $\omega \prec G \prec \iota$;

(3) *If* $\Phi \neq \omega$ *then* $\operatorname{Con} L$ *is the chain:* $\omega \prec \Phi \preceq G \prec \iota$.

PROOF: (1) Suppose that $G^* = \omega$. Then for any $x \in L$, we have $x = x^{**}$. It follows that $L = L^*$, and so we have by Theorem 5, L is simple. Thus we have $\operatorname{Con} L = \{\omega, \iota\}$. Similarly, if $G^+ = \omega$ we also obtain L is simple.

(2) Suppose that $\Phi = \omega$ and $G \neq \omega$. Then by Theorem 9 we have G is the comonolith of $\operatorname{Con} L$. By the similar method to Theorem 7, we obtain G is the monolith of $\operatorname{Con} L$. Hence, $\operatorname{Con} L$ is the chain: $\omega \prec G \prec \iota$.

(3) Suppose that $\Phi \neq \omega$. We have $[\Phi, \iota] \simeq \operatorname{Con} L/\Phi \simeq \operatorname{Con} L^\circ$ where L° is subdirectly irreducible by Corollary 4. Now if $x, y \in L^\circ$ then $x = x^\circ$ and $y = y^\circ$. Since $\Phi|_{L^\circ} = \omega$ then by (2) we have $\operatorname{Con} L^\circ$ is the chain: $\omega|_{L^\circ} \preceq G|_{L^\circ} \prec \iota|_{L^\circ}$. It follows by the congruence extension property and Theorem 2 that $[\Phi, \iota]$ is the chain: $\Phi \preceq G \prec \iota$. □

For the purpose of investigating the subdirectly irreducible members of a **CDS**-algebra, we need the following technical result.

THEOREM 11. *Let* $(L;^*, ^+, ^\circ) \in \mathbf{CDS}$ *be subdirectly irreducible. Then* L *has no 6-element chain.*

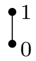
PROOF: Suppose that there exist $a, b, c, d \in L$ with $0 < a < b < c < d < 1$, then $a^* = b^* = c^* = d^* = 0$ and $a^+ = b^+ = c^+ = d^+ = 1$ by Corollary 2. It is obvious by Theorem 1(4) that $d^\circ = d$. On the one hand, if $c^\circ = c$, it follows from Theorem 6(1) and (2) that $b^\circ = d$ and $a^\circ = c$. Whence it follows the contradiction that $c^\circ \leq b^\circ$.

On the one hand, if $c^\circ = d$, we must have $b^\circ = b$. For otherwise, if $b^\circ = d$ then it follows a contradiction to Theorem 6(1); if $b^\circ = c$ then, $b^{\circ\circ} = c^\circ = d \neq b^\circ$, a contradiction. If now $a^\circ = a$, it follows a contradiction to Theorem 6(1) again. Since $a^\circ \leq b^\circ$, we must have $a^\circ = b$. Then it follows by Theorem 3 the contradiction that

$$\theta(a, b) \wedge \theta(c, d) = \theta_{lat}(a, b) \wedge \theta_{lat}(c, d) = \omega.$$ □

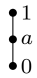
Using Theorem 5 and 6, together with Corollary 2, we now can completely characterize the subdirectly irreducible **CDS**-algebras.

THEOREM 12. *Let* $(L;^*, ^+, ^\circ) \in \mathbf{CDS}$. *There are seven non-isomorphic subdirectly irreducible members in* **CDS** *that are given by the following Hasse diagrams.*



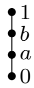
	0	1
*	1	0
+	1	0
∘	0	1

(C_1)



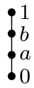
	0	a	1
*	1	0	0
+	1	1	0
∘	0	a	1

(C_2)



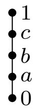
	0	a	b	1
*	1	0	0	0
+	1	1	1	0
∘	0	a	b	1

(C_3)



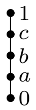
	0	a	b	1
*	1	0	0	0
+	1	1	1	0
∘	0	b	b	1

(C_4)



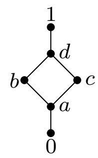
	0	a	b	c	1
*	1	0	0	0	0
+	1	1	1	1	0
∘	0	b	b	c	1

(C_5)



	0	a	b	c	1
*	1	0	0	0	0
+	1	1	1	1	0
∘	0	a	c	c	1

(C_6)



	0	a	b	c	d	1
*	1	0	0	0	0	0
+	1	1	1	1	1	0
∘	0	b	b	d	d	1

(C_7)

PROOF: We have seen that $L = [x]G \cup \{0, 1\}$ for some $x \in L \backslash \{0, 1\}$ with $|[x]G| \leq 4$. Since $|L| \leq 6$ and L has one and only one atom (coatom), using these observations and Theorem 5 and Theorem 6, we now can examine in turn each possibility for $|[x]G|$ as follows.

(1) If $|[x]G| = 0$ then, clearly L is of the form C_1.

(2) If $|[x]G| = 1$. Then L is of the form C_2.

(3) If $|[x]G| = 2$. In this case we have by Theorem 5, corollary 2 and Theorem 6 that L is of the form C_3 or C_4.

(4) If $|[x]G| = 3$. By Theorem 5, corollary 2 and Theorem 6 again that L is of the form C_5 or C_6.

(5) If $|[x]G| = 4$. Noting that L has no 6-element chain by theorem 11, by Theorem 5, corollary 2 and Theorem 6 again that L is of the form C_7. □

The lattice of subvarieties of the variety of **CDS**-algebras can be deduced from the ordered set (see Figure 1) using a classic theorem of Davey [7]. This states that **CDS** is a congruence-distributive variety generated by a finite set of finite algebras, and if the set $Si(\mathbf{CDS})$ of subdirectly irreducible algebras in **CDS** is ordered by

$$A \leqslant B \iff A \text{ is a homomorphic image of a subalgebra of } B,$$

then the lattice $\Lambda(\mathbf{CDS})$ of variety of **CDS** is isomorphic to the finite distributive lattice of down-sets of $Si(\mathbf{CDS})$. Apply this result to the variety of **CDS**-algebras, we then can obtain the lattice $\Lambda(\mathbf{CDS})$ as described in Figure 2 below, in which $\mathbf{C_i}$ denotes the (join-irreducible) subvariety generated by the subdirectly irreducible algebra C_i.

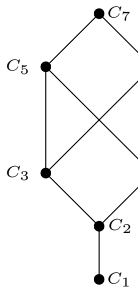


Figure 1

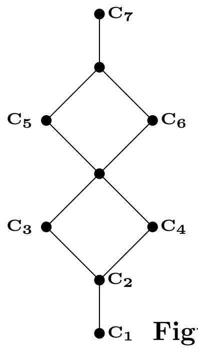


Figure 2

Acknowledgments I would like to express my appreciation deeply to the referee of this paper.

References

[1] R. Balbes and Ph. Dwinger, **Distributive Lattices**, University of Missouri Press, Missouri, 1974.

[2] T. S. Blyth and J. C. Varlet, **Ockham Algebras**, Oxford University Press, Oxford, 1994.

[3] T. Katriňák, *The structure of distributive double p-algebras. Regularity and congruences*, **Algebra Universalis** 3 (1973), pp. 238–246.

[4] T. Katriňák, *Subdirectly irreducible distributive double p-algebras*, **Algebra Universalis** 10 (1980), pp. 195–219.

[5] T. S. Blyth and Jie Fang, *Extended Ockham algebra*, **Communications in Algebra** 28(3), (2000), pp. 1271–1284.

[6] T. S. Blyth and Jie Fang, *The lattice of congruences of a subdirectly irreducible extended Ockham algebra*, **Communications in Algebra** 30(10), (2002), pp. 5023–5035.

[7] B. A. Davey, *On the lattice of subvarieties*, **Houston J. Math.** 5 (1979), pp. 183–192.

College of Mathematics and Software Science
Sichuan Normal University
Chengdu, 610066, P. R. China
e-mail: leibowang@hotmail.com