

Yehuda Schwartz
and
George Tourlakis

A PROOF THEORETIC TOOL FOR FIRST-ORDER MODAL LOGIC¹

Abstract

The authors of [11] initiated the (syntactic) proof theoretic study of the first-order modal logics M^3 and BM that were originally introduced in [5, 14, 15]. The principal tool in [11] was the “Gentzenisation” of said logics, in the paradigm of [8, 9, 16], and cut elimination. In this paper we introduce a new technique based on formula to formula mappings that we call “formulators”, which preserve provability. As applications we present both positive and negative metatheoretical results for three logics: the aforementioned M^3 ; another logic, ML^3 , that we introduce here as a first-order counterpart of the logic GL (of [2, 6, 8, 9, 16]); and for the *quantified version of GL* , QGL , obtained from classical first order logic by adding the *GL axiom-schemata* without requiring – as we do for M^3 and ML^3 – that $\Box A$ is always closed. QGL is a logic that does not allow cut elimination ([1]) – that is, its natural Gentzen equivalent does not – and this demonstrates the significance of the formulator tool; not only allows *simple* proofs, bypassing two highly non trivial steps for the logic under study – namely, Gentzenisation and cut elimination – but it also finds application to predicative modal logics, such as QGL , where cut elimination is *provably impossible*.

Among the positive results we prove that the *weak reflection* property and the *conservation* property hold for all of M^3 , ML^3 and QGL . We also establish the negative results that neither $A \rightarrow \Box A$ nor $\Box A \rightarrow A$ are provable in any of the

¹This research was partially supported by NSERC grant No. 8250.

three logics. Our positive and negative results for QGL are, as far as we know, new in the literature.

Keywords: Modal logic, GL, first order logic, proof theory, cut elimination, reflection property, quantified modal logic, QGL.

1. Introduction

Two natural 1st-order modal logics, M^3 and BM, were introduced in [14, 15, 5] where the modal “box”, \Box , faithfully simulated the classical metatheoretical predicate “is a theorem from”, \vdash . This faithful simulation was expressed by the *conservation theorem* of loc. cit., that is, “a classical formula A is classically provable from premises Γ iff $\Box A$ is modally provable from premises Γ and $\Box \Gamma$.²” This result as well as the related *weak reflection* that, for any modal 1st-order formula A , if $\Box A$ is modally provable then so is A , were proved semantically using appropriate Kripke models.

Cut elimination is a standard and powerful proof theoretic tool via which [11] demonstrated the weak reflection property and the conservatism of M^3 and BM *syntactically*, after showing first that a certain Gentzen-style proxy, GTKS, of these two logics admits cut elimination.

However, not every proof theoretic result requires cut elimination. In fact, we introduce here a new tool for proving conservation and weak reflection. We demonstrate the power of this tool on two logics that we have devised, the M^3 of [14, 15], and a 1st-order counterpart of the well-known provability logic GL that we call ML^3 .³ Both these logics allow cut elimination (i.e., their Gentzen equivalents do). For cut-eliminability of M^3 see [11] while the case of ML^3 is treated in the first author’s PhD thesis, [12]. Remarkably, the tool introduced here is applicable also to QGL, a first-order extension of GL that does not require that $\Box A$ be a sentence for all A . What is remarkable is that our new tool can prove both positive (support of both weak reflection and conservation theorem) and negative results (failure of both strong reflection and necessitation) for QGL, and we can so prove despite the fact that this logic, *provably* ([1]), does not admit cut elimination.

This tool offers more than *power*: It also offers *convenience*.

² $\Box \Gamma = \{\Box X : X \in \Gamma\}$.

³The former is obtained from the latter by replacing the $\Box A \rightarrow \Box \Box A$ axiom with Löb’s axiom.

If we understand as “power” its ability to obtain positive results such as weak reflection in QGL, and establish the failure of both strong reflection and strong necessitation in said logic where cut elimination is unavailable, then “convenience” is manifested in those cases, such as M^3 and ML^3 , where it offers *simple* proofs, bypassing two highly non trivial steps for the logic under study – namely, Gentzenisation and cut elimination.

As far as we know, the positive and negative results regarding QGL are new in the literature.

This tool involves a class of mappings (which we call *formula translators* or, for short, *formulators*), from the set of well-formed modal formulae to itself, that preserve proofs. More precisely, we demonstrate that if $\Gamma \vdash A$ in any of the above three logics, then for any well-chosen formulator \mathfrak{F} , we have $\mathfrak{F}(\Gamma) \vdash \mathfrak{F}(A)$.

2. Prime Formulae, Substitutions and Tautologies

We follow [10, 11] in the choice of connectives used toward building formulae: $\rightarrow, \perp, \forall$ and \Box . Our 1st-order modal alphabet contains predicates, ϕ, ψ (with or without primes or subscripts) but neither constant nor function symbols. There is no distinguished⁴ predicate called “equality”. The detailed rules of formula formation are trivial variations of the norm and will be omitted. We note however that as is the case in the foregoing two references, we employ two types of object variables: *free*, denoted by a, b, c with or without primes or subscripts, and *bound*, denoted by x, y, z with or without primes or subscripts. All formulae of the type $\Box A$ are closed by virtue of the syntactic construction: All the free variables of A are replaced by new bound variables and then \Box is concatenated to the left of the so altered A . Thus “ $\Box A$ ” is metanotation for the here described string operation, just as in Bourbaki ([3]) the expressions “ $\tau_x A$ ” and “ $\exists x A$ ” are metanotations that hide the details of the actual syntax.

DEFINITION 2.1. (Prime Formulae) A well formed modal formula (wfmf), A , is *prime* if it is one of the following:

⁴The distinction is normally expressed by a special symbol and a number of logical axioms governing the behaviour of equality.

1. Any atomic formula other than \perp
2. $\forall xB$ for some wfmf B .
3. $\Box B$ for some wfmf B . ■

We shall use lowercase p, q, r in order to denote prime formulae. Using prime formulae we can abstract the modal and 1st-order structure of a wfmf thus viewing it as a propositional (Boolean) formula where the participating prime formulae play the role of Boolean variables.

DEFINITION 2.2. Let A be a well formed modal formula, then we define $P_r(A)$ as the set of all the prime sub-formulae of A . ■

NOTATION 2.3

1. Let A be a formula and let $\{p_1, \dots, p_n\}$ be a set of n prime formulae such that $P_r(A) \subseteq \{p_1, \dots, p_n\}$, then we may write A as $A(p_1, \dots, p_n)$.⁵
2. Let A be $A(\vec{p})$ then, $A[p_i := B]$, where B is *any* formula, denotes the result of simultaneously replacing each occurrence of p_i in A with B .⁶ More precisely:

$$A[p := B] = \begin{cases} B & \text{if } A \text{ is } p \\ A & \text{if } A \text{ is } \perp \text{ or } q \neq p \\ C[p := B] \rightarrow D[p := B] & \text{if } A \text{ is } C \rightarrow D \end{cases}$$

NOTE. In this paper we employ the metanotation “ $[\dots := \dots]$ ” also to indicate the (everywhere in an expression) replacement of the object variable “ \dots ” by the expression “ \dots ”. In all cases the metaexpression “ $[\dots := \dots]$ ” has the highest priority of application (least scope).

DEFINITION 2.4. (Propositional Valuation) A valuation V is a function from the set of all prime formulae to the set of two distinct symbols $\{F, T\}$ that satisfies

1. $V(\perp) = F$
2. $V(A \rightarrow B) = \text{if } V(A) = T \wedge V(B) = F \text{ then } F \text{ else } T$.

⁵Thus, it is implicit in the notation $A(p_1, \dots, p_n)$ that $p_i \neq p_j$ for $1 \leq i \neq j \leq n$.

⁶Thus, if $p_i \notin P_r(A)$ then $A[p_i := B] = A$.

We pronounce “ T ” and “ F ” “true” and “false” respectively ■

By the above, we are free to choose $V(p)$ in any way we please for prime p . Moreover a trivial induction on Boolean formulae shows that our definition of V on prime formulae uniquely determines V on all formulae A . As it is standard practise, we define *tautologies* as those Boolean formulae A that satisfy $V(A) = T$ for all V .

The proof of the following three propositions is done by a straightforward induction on the construction of a propositional formula, and thus will be omitted.

PROPOSITION 2.5.

1. Let A be $A(\vec{p})$ and p, q prime formulae such that $q \notin P_r(A)$, then $A[p := q][q := p]$ is A .
2. Let A be $A(\vec{p})$, p and q prime formulae, $B = B(r_1, \dots, r_m)$, then if $q \notin P_r(A) \cup P_r(B)$ then $A[p := B] = A[p := B[r := q]][q := r]$.

PROPOSITION 2.6. Let A be a formula and V_1, V_2 be two valuations such that $V_1 = V_2$ on $P_r(A)$, then $V_1(A) = V_2(A)$.

PROPOSITION 2.7. Let A be $A(\vec{p})$, p a prime formula, B any formula, and let A' be $A[p := B]$, then if V is a valuation such that $V(p) = V(B)$ then $V(A) = V(A')$.

LEMMA 2.8. If $A = A(\vec{p})$ is a tautology, then so is $A[p := B]$ where $p \notin P_r(B)$.

PROOF. Let A' stand for $A[p := B]$. Let V be any valuation, and assume that $V(p) \neq V(B)$. By hypothesis, $V(A) = T$ and so by 2.7, $V(A') = V(A) = T$.

If $V(p) \neq V(B)$ then examine \bar{V} , which is the valuation that is equal to V on all prime formulae *except for* p on which it is different (so $\bar{V}(p) = \bar{V}(B)$). Like before, $\bar{V}(A') = \bar{V}(A) = T$, but since $p \notin P_r(A')$ then \bar{V} agrees with V on $P_r(A')$ and so, by 2.6, $V(A') = \bar{V}(A') = T$. ■

COROLLARY 2.9. If A is $A(\vec{p})$ and is a tautology, then so is $A[p := B]$ where B is any formula.

PROOF. Let A' stand for $A[p := B]$. If $p \notin P_r(B)$ then we are done by 2.8. Otherwise, examine $B' = B[p := q]$ where $q \notin P_r(B) \cup P_r(A)$ and let A''

be $A[p := B'] = A[p := B[p := q]]$. Now, by 2.8 again, A'' is a tautology. Similarly, $\hat{A} = A''[q := p]$ is also a tautology. But $\hat{A} = A[p := B[p := q]][q := p]$ and so, by 2.5, $\hat{A} = A[p := B] = A'$ and we are done. ■

DEFINITION 2.10. (Simultaneous Substitution) Given formulae $A = A(p_1, \dots, p_n)$ and B_1, \dots, B_n . Let q_1, \dots, q_n be n distinct prime formulae not in $P_r(A) \cup \bigcup_{i=1}^n P_r(B_i)$. Then we define the *simultaneous substitution* $A(B_1, \dots, B_n)$ (or $A[p_1, \dots, p_n := B_1, \dots, B_n]$) as $A[p_1 := q_1][p_2 := q_2] \dots [p_n := q_n][q_1 := B_1][q_2 := B_2] \dots [q_n := B_n]$. ■

Note that the operation of simultaneous substitution is well-defined, that is, it does not depend on the order of the substitution as is verifiable by induction on the construction of A . Also note that this definition covers the case where we wish to only (simultaneously) substitute a subset of the p_k – for example, the result of only substituting B_1 for p_i and B_2 for p_j ($A[p_i, p_j := B_1, B_2]$) can be written as $A(p_1, \dots, B_1, \dots, B_2, \dots, p_n)$; that is, we assume $B_k = p_k$ for all $k \neq i, j$.

This leads us to the next corollary:

COROLLARY 2.11. *If $A = A(p_1, \dots, p_n)$ is a tautology, then this is also true of $A(B_1, \dots, B_n)$, where B_i is a formula for $i = 1, \dots, n$.*

PROOF. By a repeated use of 2.9. ■

3. The 1st-order modal logics M^3 and ML^3

The first-order modal logic M^3 below was originally introduced and studied *model-theoretically* (along with the logic BM) in [14, 15, 5]. Its *syntactic* proof theory was first developed in [11] via cut elimination that we proved holds in a proxy Gentzen-style logic GTKS.

DEFINITION 3.1. (Axioms and Rules of Inference for M^3) The set of logical axioms of M^3 is $\Lambda \cup \Box\Lambda$, where Λ consists of all instances of the following basic schemata:

- (1) All tautologies
- (2) $\forall x A[x] \rightarrow A[a]$
- (3) $A[a] \rightarrow \forall x A[x]$, provided a does not occur in A .
- (4) $\forall x (A \rightarrow B) \rightarrow \forall x A \rightarrow \forall x B$
- (5) $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$

- (6) $\Box A \rightarrow \Box \Box A$
- (7) $\Box A \rightarrow \Box \forall x A$

There are two *primary rules of inference*: Modus ponens (MP) “if A and $A \rightarrow B$, then infer B ”, and generalisation (Gen) “if A , then infer $\forall x A$ ”. Derivability in M^3 is denoted by $\Gamma \vdash_{M^3} A$, that is, A is derived from hypotheses Γ . ■

A closely related 1st-order logic, ML^3 is defined below by, essentially, replacing axiom (6) above by Löb’s axiom (6’). This logic that is closely related to the propositional provability logic GL was first introduced in [12], along with a proxy Gentzen-style system GLTS that admits cut elimination. The latter was the main tool, once again, to syntactically develop the proof theory of ML^3 .

As we indicated in the introduction section, we will revisit in this paper the proof theory of both M^3 and ML^3 via the formulator approach.

DEFINITION 3.2. (Axioms and Rules of Inference for ML^3) The set of logical axioms of ML^3 is $\Lambda \cup \Box \Lambda \cup \Box \Box \Lambda$,² where Λ consists of all instances of the following basic schemata:

- (1’) All tautologies
- (2’) $\forall x A[x] \rightarrow A[a]$
- (3’) $A[a] \rightarrow \forall x A[x]$, provided a does not occur A .
- (4’) $\forall x (A \rightarrow B) \rightarrow \forall x A \rightarrow \forall x B$
- (5’) $\Box (A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$
- (6’) $\Box (\Box A \rightarrow A) \rightarrow \Box A$
- (7’) $\Box A \rightarrow \Box \forall x A$

There are two *primary rules of inference*: modus ponens (MP) and generalisation. Derivability in ML^3 is denoted by $\Gamma \vdash_{ML^3} A$, that is, A is derived from hypotheses Γ . ■

NOTE. As is well known, axiom groups (1) and (1’) allow *proof by tautological implication*, that is, if $A_1, A_2, \dots, A_n \models_{taut} X$ – an abbreviation of “ $A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n \rightarrow X$ is a tautology” – then X is derivable (syntactically) from the hypotheses A_1, A_2, \dots

²The presence of $\Box \Box \Lambda$ helps in proving the derivability of $\Box A \rightarrow \Box \Box A$ in ML^3 .

In what follows we will use the abbreviation “ $A \wedge B$ ” for “ $(A \rightarrow (B \rightarrow \perp)) \rightarrow \perp$ ”.

REMARK 3.3. (Weak Necessitation (WN)) It is easy to show (see [14]) that if $\Gamma \vdash A$, then $\Gamma, \Box \Gamma \vdash \Box A$, where “ \vdash ” is one of \vdash_{M^3} or \vdash_{ML^3} . On the other hand, “strong necessitation”, $A \rightarrow \Box A$, is not provable in either M^3 or ML^3 as we prove in Section 7. The “trick” of obtaining necessitation as a derived rule, “hiding it” in effect in the axiom schemata, is well-known (cf. [13]) and goes back to the analogous trick of hiding the generalisation rules in the axioms of classical first-order logic, as in [4]. Nevertheless, while we prefer to have WN as a *derived rule*, we have allowed “strong” generalisation as a primary rule in the style of [7]. This allows a natural formulation of the “conservation result” for M^3 and ML^3 . ■

4. Formulators

To allow wide applicability of the formulator tool beyond M^3 and ML^3 , we define them in the broader context where the free variables of $\Box A$ are a subset of those of A . This makes formulators also relevant to logics such as QGL.

DEFINITION 4.1. (Formulators) A *formula translator* or a *formulator* is a function, \mathfrak{F} , from the set of wfmf to itself such that:

1. $\mathfrak{F}(A) = A$ for every *atomic* formula A .
2. $\mathfrak{F}(A \rightarrow B) = \mathfrak{F}(A) \rightarrow \mathfrak{F}(B)$ for all formulae A, B .
3. $\mathfrak{F}(\forall x A[a := x]) = \forall x B[a := x]$, where $B = \mathfrak{F}(A[a])$.
4. The free variables of $\mathfrak{F}(\Box A)$ are among those of $\Box A$. ■

REMARK 4.2. (1) As was carefully defined in [11], the metalogical abbreviation “ $\forall(x, a)A$ ” or simply “ $\forall x A$ ” if the free variable a is understood from the context, stands for the expression $\forall x A[a := x]$ formed by first replacing all the a by the bound variable x and then by prepending the string $\forall x$ to the so obtained expression. Thus the import of 3 above is that a formulator must commute with the universal quantifier.

(2) We note that this (recursive) definition entails that any two formulators which agree on the set of *boxed formulae* – that is, formulae of the form $\Box A$ – must be identical. In other words, we need only to define the operation of a given formulator on boxed formulae in order to know the

nature of its operation on *all* formulae. This definition also means that for any *classical* formula A and any formulator \mathfrak{F} we must have $\mathfrak{F}(A) = A$.

(3) Condition 4 in the definition ensures (by a straightforward induction on formulae) that the free variables of $\mathfrak{F}(A)$ are among those in A . This, in particular, is relevant to ensuring that the image of an instance of axiom (schema) 3 under \mathfrak{F} of, for example, M^3 is an instance of the same axiom schema. ■

LEMMA 4.3. *For any wfmf $A = A(\vec{p})$ and for any formulator \mathfrak{F} , the identity $\mathfrak{F}(A) = A(\mathfrak{F}(p_1), \dots, \mathfrak{F}(p_n))$ holds.⁷*

PROOF. Induction on the construction of A (as a propositional formula).

1. If A is \perp , then no p_i occurs in A , thus the claim is vacuously satisfied.
2. If $A = p$ (prime p), then $\mathfrak{F}(A) = \mathfrak{F}(p) = p[p := \mathfrak{F}(p)] = A[p := \mathfrak{F}(p)]$ as needed.
3. If A is $B \rightarrow C$, then $\mathfrak{F}((B \rightarrow C)(p_1, \dots, p_k)) = \mathfrak{F}(B(p_1, \dots, p_k)) \rightarrow \mathfrak{F}(C(p_1, \dots, p_k)) \stackrel{\text{I.H.}}{=} B(\mathfrak{F}(p_1), \dots, \mathfrak{F}(p_n)) \rightarrow C(\mathfrak{F}(p_1), \dots, \mathfrak{F}(p_n)) = A(\mathfrak{F}(p_1), \dots, \mathfrak{F}(p_n))$. ■

To fix ideas in what follows, we will adopt a particular form of the “quantified GL” logic, QGL: Its axiom set has two parts, Λ_{QGL} and $\Box\Lambda_{QGL}$ where Λ_{QGL} consists of the instances of the axiom schemata (1)–(6) of 3.1 and (6′) of 3.2. Thus, necessitation is a derived rule WN, and the two primitive rules are the same as those of M^3 .

DEFINITION 4.4. In what follows we let CL stand, *somewhat ambiguously*, for the classical logic based on the axiom set Λ_{CL} that consists of all the instances of (1)–(4) in 3.1, and whose primary rules are those of M^3 . The benign ambiguity is in regards to the underlying first-order language: In the case of M^3 and ML^3 the common first-order *modal* language is as defined in this paper, with the elaborate syntax for the *abbreviation* “ $\Box A$ ” that renders this formula *closed*. For the language of QGL, there is a deviation: $\Box A$ is not an abbreviation at all, rather if the expression named A is a formula, then so is the one that is prefixed with \Box . Moreover, the free variables of $\Box A$ in the language of QGL are precisely those of A . ■

LEMMA 4.5. *If $A \in \Lambda_{CL}$ then $\mathfrak{F}(A) \in \Lambda_{CL}$.*

⁷I.e., a formulator may have arbitrary (non prime) result on prime formulae, but commutes with Boolean connectives.

PROOF.

1. A is a tautology. Let $A = A(\vec{p})$ then by 4.3, $\mathfrak{F}(A) = A(\mathfrak{F}(p_1), \dots, \mathfrak{F}(p_n))$, and by 2.11, $\mathfrak{F}(A)$ is also a tautology.
2. $A = \forall x B[a := x] \rightarrow B[a]$. $\mathfrak{F}(A) = \forall x \mathfrak{F}(B[a])[a := x] \rightarrow \mathfrak{F}(B[a])$, which is an axiom of the same form.
3. $A = B[a] \rightarrow \forall x B[x]$, where a does not occur in B . $\mathfrak{F}(A) = \mathfrak{F}(B[a]) \rightarrow \forall x \mathfrak{F}(B[a])[a := x]$, which is in Λ_{CL} since a does not occur in $\mathfrak{F}(B)$.⁸
4. $A = \forall x (B \rightarrow C) \rightarrow (\forall x B \rightarrow \forall x C)$. $\mathfrak{F}(A) = \forall x (\mathfrak{F}(B) \rightarrow \mathfrak{F}(C)) \rightarrow (\forall x \mathfrak{F}(B) \rightarrow \forall x \mathfrak{F}(C))$, which is in Λ_{CL} . ■

THEOREM 4.6. *If $\Gamma \vdash_{CL} A$ and \mathfrak{F} is a formulator then $\mathfrak{F}(\Gamma) \vdash_{CL} \mathfrak{F}(A)$, where $\mathfrak{F}(\Gamma) = \{\mathfrak{F}(A) : A \in \Gamma\}$.*

PROOF. Let us simply write \vdash_{CL} as \vdash . By induction on (Hilbert style) proofs in CL.

- I. $A \in \Gamma$. Then $\mathfrak{F}(A) \in \mathfrak{F}(\Gamma)$ and thus $\mathfrak{F}(A) \vdash \mathfrak{F}(\Gamma)$.
- II. $A \in \Lambda_{CL}$. Then, by 4.5, $\mathfrak{F}(A) \in \Lambda_{CL}$. Hence

$$\vdash \mathfrak{F}(A) \tag{*}$$

Thus, $\mathfrak{F}(A) \vdash \mathfrak{F}(\Gamma)$.

- III. $\Gamma \vdash B \rightarrow A$ and $\Gamma \vdash B$. Then by I.H. $\mathfrak{F}(\Gamma) \vdash \mathfrak{F}(B \rightarrow A)$ but, by Definition 4.1, case 3, this means that $\mathfrak{F}(\Gamma) \vdash \mathfrak{F}(B) \rightarrow \mathfrak{F}(A)$. One more invocation of the I.H. yields $\mathfrak{F}(\Gamma) \vdash \mathfrak{F}(B)$, thus, using MP, we get $\mathfrak{F}(\Gamma) \vdash \mathfrak{F}(A)$.
- IV. $A = \forall x B(x)$ and $\Gamma \vdash B[a]$ (application of generalisation). By the I.H., $\mathfrak{F}(\Gamma) \vdash \mathfrak{F}(B[a])$. By generalisation we get $\mathfrak{F}(\Gamma) \vdash \forall x \mathfrak{F}(B[a])[a := x]$. By Definition 4.1, case 3, this is the same as $\mathfrak{F}(\Gamma) \vdash \mathfrak{F}(\forall x (B[a := x]))$. ■

We are interested in formulators that preserve theorems of interesting modal logics. Thus we define,

DEFINITION 4.7. (Conservative Formulators) A formulator \mathfrak{F} over a first-order modal language is *conservative for some logic L over said language* iff whenever $\Gamma \vdash_L A$ we also have $\mathfrak{F}(\Gamma) \vdash_L \mathfrak{F}(A)$. ■

⁸We can show, using induction on the construction of a wfmf, that for every formula A , the list of free variables of $\mathfrak{F}(A)$ is contained in that of A . Pivotal in this induction is case 4 in the formulator definition (4.1).

We may readily reuse the proof of 4.6, this time for logic L rather than CL toward the following *criterion of conservatism of \mathfrak{F} for a logic L* :

PROPOSITION 4.8. *Let L be a first-order modal logic with axioms $\Lambda_L = \Lambda_{CL} \cup \Omega_L$, and with only modus ponens and generalisation as primary rules. Given a formulator \mathfrak{F} , it is sufficient that $\vdash_L \mathfrak{F}(A)$, for all $A \in \Omega_L$, in order that \mathfrak{F} be conservative for L .*

PROOF. Same as the proof of 4.6, replacing CL and its axioms by L and its axioms. Note that the “hard case” was II, toward establishing assertion (*). The remaining cases in the proof trivially fended for themselves and can do identically so in the case of L . ■

It is part of the folklore of modal (propositional) logic that the Löb axiom 6' in 3.2 can simulate axiom 6. Nevertheless, in what follows we employ a modified definition of ML^3 , where we add 6 explicitly. This apparent redundancy allows us as a trade-off to simplify the axiom set of the logic in a different direction: In the presence of axiom 6, the full set of axioms now is $\Lambda_{ML^3} \cup \Box\Box\Lambda_{ML^3}$. *That is, the $\Box\Box\Lambda_{ML^3}$ part is discarded.*⁹

5. Some Useful Formulators

In this section we will present a number of specific formulators with specified behaviour on boxed formulae.

The Strong-Box Formulators

DEFINITION 5.1. We let \mathfrak{S} be the formulator that, for every formula A , satisfies $\mathfrak{S}(\Box A) = \Box A \wedge \forall A$, and we let \mathfrak{S}_0 be the formulator that, for every formula A , satisfies $\mathfrak{S}_0(\Box A) = \Box A \wedge A$, where $\forall A$ denotes the universal closure of A , and “ \wedge ” is the “and” metasymbol defined from \rightarrow and \perp in the standard manner.

LEMMA 5.2. *\mathfrak{S} is conservative for each of M^3 and ML^3 .*

PROOF. By Proposition 4.8, we only need to show that $\vdash_X \mathfrak{S}(A)$, for all $A \in \Omega_X$, for each case $X \in \{M^3, ML^3\}$.

⁹The role of this part was to actually help the derivation of 6 from 6'! Conversely, every formula in $\Box\Box\Lambda_{ML^3}$ is derivable in the new formulation of ML^3 by using axiom 6.

1. $A = \Box(B \rightarrow C) \rightarrow \Box B \rightarrow \Box C$. Then $\mathfrak{S}(A) = \Box(B \rightarrow C) \wedge \forall(B \rightarrow C) \rightarrow (\Box B \wedge \forall B) \rightarrow (\Box C \wedge \forall C)$. To show $\vdash_X \mathfrak{S}(A)$, for any of the above mentioned X , is to show $\vdash_X \Box(B \rightarrow C) \wedge \forall(B \rightarrow C) \wedge \Box B \wedge \forall B \rightarrow \Box C \wedge \forall C$. Indeed, combining the facts that $\vdash_X \Box(B \rightarrow C) \wedge \Box B \rightarrow \Box C$ and $\vdash_X \forall B \wedge \forall(B \rightarrow C) \rightarrow \forall C$ we can deduce that $\vdash_X \mathfrak{S}(A)$ by tautological implication.
2. $A = \Box B \rightarrow \Box \Box B$. Then $\mathfrak{S}(A) = \Box B \wedge \forall B \rightarrow \Box \Box B \wedge \forall \Box B$. Using the facts that $\vdash_X \Box B \rightarrow \forall \Box B$ – axiom 3 of 3.1 since $\Box B$ is closed in the language of X – and $\vdash_X \Box B \rightarrow \Box \Box B$, for both X as above, yields that $\vdash_X \mathfrak{S}(A)$.
3. $A = \Box B \rightarrow \Box \forall x B$. Then $\mathfrak{S}(A) = \Box B \wedge \forall B \rightarrow \Box \forall x B \wedge \forall B$, which is derivable in X by tautological implication, since $\Box B \rightarrow \Box \forall x B \in \Lambda_X$.
4. $A \in \Box \Lambda_X$. Then $A = \Box B$ for some $B \in \Lambda_X$. Now, $\mathfrak{S}(A) = \Box B \wedge \forall B$, which is provable in X since $\Box B \in \Box \Lambda_X$ and $\forall B$ is provable through generalisation(s) on $B \in \Lambda_X$.
5. *Löb's axiom; applicable only to ML^3* : $A = \Box(\Box B \rightarrow B) \rightarrow \Box B$. Then $\mathfrak{S}(A) = \Box(\Box B \rightarrow B) \wedge \forall(\Box B \rightarrow B) \rightarrow \Box B \wedge \forall B$, which is provable in ML^3 because a) through c) are derivable
 - a) $\Box B \rightarrow \forall \Box B$ ¹⁰ (axiom 3)
 - b) $\Box(\Box B \rightarrow B) \rightarrow \Box B$
 - c) $\forall(\Box B \rightarrow B) \rightarrow \forall \Box B \rightarrow \forall B$ (via axiom 4)
 thus, a) and b) yield via tautological implication
 - d) $\Box(\Box B \rightarrow B) \rightarrow \forall \Box B$
 and in the same manner we get
 - e) $\Box(\Box B \rightarrow B) \rightarrow \forall(\Box B \rightarrow B) \rightarrow \forall B$ from c) and d)
 Finally, in the same manner, b) and e) derive $\Box(\Box B \rightarrow B) \wedge \forall(\Box B \rightarrow B) \rightarrow \Box B \wedge \forall B$.
6. A is $\Box(\Box(\Box B \rightarrow B) \rightarrow \Box B)$. We are done from the fact that $A \in \Box \Lambda_{ML^3}$ and that $\forall(\Box(\Box B \rightarrow B) \rightarrow \Box B)$ is provable in ML^3 since $\Box(\Box B \rightarrow B) \rightarrow \Box B$ is closed in the underlying language. ■

COROLLARY 5.3. \mathfrak{S}_0 is conservative for QGL.

PROOF. As the $\Box A \rightarrow \Box \forall x A$ axiom does not apply to QGL, and the other cases being straightforward, we need only modify the proof of item 5 in the above proof as follows:

¹⁰In the language of ML^3 $\Box A$ is closed.

Let $A = \Box(\Box B \rightarrow B) \rightarrow \Box B$. Then $\mathfrak{S}_0(A) = \Box(\Box B \rightarrow B) \wedge (\Box B \rightarrow B) \rightarrow \Box B \wedge B$, which is provable in QGL because a) (axiom) and b) (tautologically from a) are derivable.

- a) $\Box(\Box B \rightarrow B) \rightarrow \Box B$
- b) $\Box(\Box B \rightarrow B) \wedge (\Box B \rightarrow B) \rightarrow \Box B \wedge (\Box B \rightarrow B)$

Also,

- c) $\Box B \wedge (\Box B \rightarrow B) \rightarrow B$ (tautology),
and, from c by tautological implication,
- d) $\Box B \wedge (\Box B \rightarrow B) \rightarrow \Box B \wedge B$

d) and b) derive $\mathfrak{S}_0(A)$ by tautological implication. ■

The Characteristic Formulator

Intuitively, \perp is the formal counterpart for “false”. Our alphabet has no formal counterpart for “true”, but the formula $\perp \rightarrow \perp$ is one possible way to effect such formalisation. We shall use the metasymbol \top to stand for $\perp \rightarrow \perp$.¹¹ Now we define

DEFINITION 5.4. Let Δ be a set of wfmf’s, then we define the *characteristic formulator of Δ* , denoted by \mathcal{X}_Δ , as the formulator whose operation on boxed formulae is as follows:

$$\mathcal{X}_\Delta(\Box A) = \begin{cases} \top & \text{If } \Delta \vdash \Box A \\ \perp & \text{If } \Delta \not\vdash \Box A \end{cases}$$

NOTE. When $\Delta = \{A\}$ for some wfmf A , we will allow ourselves to abuse the notation and use \mathcal{X}_A instead of $\mathcal{X}_{\{A\}}$.

LEMMA 5.5. \mathcal{X}_Δ , for any Δ , is conservative for each of M^3 , ML^3 and QGL .

PROOF. By 4.8 we only need to examine the following cases, where $X \in \{M^3, ML^3, QGL\}$:

- (1) $A = \Box(B \rightarrow C) \wedge \Box B \rightarrow \Box C$. If $\Delta \vdash_X \Box C$ then $\mathcal{X}_\Delta(\Box C) = \top$ and $\mathcal{X}_\Delta(\Box(B \rightarrow C) \wedge \Box B \rightarrow \Box C) = \mathcal{X}_\Delta(\Box(B \rightarrow C) \wedge \Box B) \rightarrow \top$, which is provable in X .

¹¹Of course, $\top \in \Lambda_{CL}$.

If $\Delta \not\vdash_X \Box C$, then we cannot have $\Delta \vdash_X \Box(B \rightarrow C) \wedge \Box B$ since otherwise we would have $\Delta \vdash_X \Box B \rightarrow \Box C$ and $\Delta \vdash_X \Box B$ contradicting the hypothesis. Thus assume, without loss of generality, that $\Delta \not\vdash_X \Box B$, then $\mathcal{X}_\Delta(\Box B) = \perp$, and $\mathcal{X}_\Delta(A) = \mathcal{X}_\Delta(\Box(B \rightarrow C)) \wedge \perp \rightarrow \perp$, which is provable in X .

- (2) $A = \Box B \rightarrow \Box \Box B$. If $\Delta \not\vdash_X \Box B$, then $\mathcal{X}_\Delta(A) = \perp \rightarrow \mathcal{X}_\Delta(\Box \Box B)$, which is provable in X . If, on the other hand, $\Delta \vdash_X \Box B$, then also $\Delta \vdash_X \Box \Box B$, by modus ponens and axiom 6 of 3.1. Thus $\mathcal{X}_\Delta(A) = \top \rightarrow \top$, which is also provable.
- (3) $A = \Box(\Box B \rightarrow B) \rightarrow \Box B$. Similarly (*this case is not relevant to M^3*).
- (4) For $X \in \{M^3, ML^3\}$, the case $A = \Box B \rightarrow \Box \forall x B$ presents no difficulties. *This case is not relevant to QGL.*
- (5) $A \in \Box \Lambda_X$. Let $A = \Box B$. But $\vdash_X \Box B$; hence $\Delta \vdash_X \Box B$. Thus, $\mathcal{X}_\Delta(A) = \top$ which is provable in X .
- (6) A is $\Box(\Box(\Box B \rightarrow B) \rightarrow \Box B)$. Similar to the previous item (*this case is not relevant to M^3*). ■

The next result is a good example of the power of the characteristic formulator.

PROPOSITION 5.6. *Let Σ, Λ be sets of classical formulae, and suppose that $\Sigma, \Box \Delta \vdash \bigvee \Box \Omega \vee \bigvee \Lambda$ then $\Sigma \vdash \bigvee \Lambda$ or there is some $B \in \Omega$ such that $\Box \Delta \vdash \Box B$, where deducibility \vdash is that of any one of M^3 , ML^3 , or QGL.*

PROOF. Assume that $\Box \Delta \not\vdash \Box B$, for all $B \in \Omega$, then, by definition, $\mathcal{X}_{\Box \Delta}(\Box B) = \perp$, for all $B \in \Omega$. Also, since clearly $\Box \Delta \vdash \Box C$ for all $C \in \Delta$, we have that $\mathcal{X}_{\Box \Delta}(\Box C) = \top$, for all such C 's. Thus, by 5.5 and the fact that formulators commute with Boolean connectives, we know that $\mathcal{X}_{\Box \Delta}(\Box \Delta), \mathcal{X}_{\Box \Delta}(\Sigma) \vdash \bigvee \mathcal{X}_{\Box \Delta}(\Box \Omega) \vee \bigvee \mathcal{X}_{\Box \Delta}(\Lambda)$, or (since Σ, Λ are classical) $\top, \Sigma \vdash \perp \vee \bigvee \Lambda$ which is the same as $\Sigma \vdash \bigvee \Lambda$. ■

6. CT and WR for M^3 , ML^3 and QGL

Let us first prove weak reflection (WR) for M^3 , ML^3 and QGL.

THEOREM 6.1. (WR for M^3 , ML^3 and QGL) *If $\Gamma, \Box \Gamma \vdash \Box A$ where Γ is classical, then $\Gamma, \Box \Gamma \vdash A$ as well.*

NOTE. We do *not* require A to be classical.

PROOF. We show first for M^3 and ML^3 . Using 5.2, we obtain $\mathfrak{S}(\Gamma), \mathfrak{S}(\Box\Gamma) \vdash \mathfrak{S}(\Box A)$, and since Γ is classical, this means $\Gamma, \forall\Gamma, \Box\Gamma \vdash \Box A \wedge \forall A$. In particular, it means that $\Gamma, \forall\Gamma, \Box\Gamma \vdash \forall A$ and since, for every formula B , B and $\forall B$ are inter-derivable in both M^3 and ML^3 , we obtain $\Gamma, \Box\Gamma \vdash A$.

The QGL case is almost identical and uses \mathfrak{S}_0 and 5.3 instead of \mathfrak{S} : We have $\mathfrak{S}_0(\Gamma), \mathfrak{S}_0(\Box\Gamma) = \Gamma, \Box\Gamma$ since one may split disjunctions at the left of \vdash . Thus, we have $\Gamma, \Box\Gamma \vdash \Box A \wedge A$, hence $\Gamma, \Box\Gamma \vdash A$ ■

In the next proposition and lemma we use the characteristic formulor of $\{\perp\}$, \mathcal{X}_\perp . Note that since $\perp \vdash X$ for every wfmf X , then $\mathcal{X}_\perp(\Box A) = \top$ for every wfmf A .

PROPOSITION 6.2. *For every wfmf A , $\mathcal{X}_\perp(A)$ is classical.*

PROOF. Induction on the complexity of A . ■

LEMMA 6.3. *If $\Gamma \vdash_X A$ then we have a classical proof of $\mathcal{X}_\perp(\Gamma) \vdash_X \mathcal{X}_\perp(A)$, where $X \in \{M^3, ML^3, QGL\}$.*

NOTE. Of course, a classical proof in X is a classical proof in CL.

PROOF. We will show by induction on theorems of X that we have a classical proof of $\mathcal{X}_\perp(\Gamma) \vdash_X \mathcal{X}_\perp(A)$.

1. $A \in \Gamma$. Then since, by the previous proposition, $\mathcal{X}_\perp(A)$ and $\mathcal{X}_\perp(\Gamma)$ are classical and since $\mathcal{X}_\perp(A) \in \mathcal{X}_\perp(\Gamma)$ we get a classical proof of $\mathcal{X}_\perp(\Gamma) \vdash \mathcal{X}_\perp(A)$.
2. $A \in \Lambda_X$. By 4.5, if A is not one of the modal axioms, then $\mathcal{X}_\perp(A) \in \Lambda_{CL}$. If A is one of the modal axioms then $\mathcal{X}_\perp(A)$ is a classical tautology since these axioms have the form $\Box B \rightarrow \Box C$ or $\Box B \rightarrow \Box C \rightarrow \Box D$.
3. $A \in \Box\Lambda_X$. Then, since A is boxed, $\mathcal{X}_\perp(A) = \top$ which is classically provable in X .
4. A was obtained from $B \rightarrow A$ and B using modus ponens. Then, by the I.H., we have a classical proof of $\mathcal{X}_\perp(B \rightarrow A)$, which is equal to $\mathcal{X}_\perp(B) \rightarrow \mathcal{X}_\perp(A)$, and a classical proof of $\mathcal{X}_\perp(B)$ and now using modus ponens we get a classical proof of $\mathcal{X}_\perp(A)$.
5. $A = \forall x B$ was obtained from B using generalisation. Then, by the I.H., we have a classical proof of $\mathcal{X}_\perp(B)$ and using generalisation we get a classical proof of $\forall x \mathcal{X}_\perp(B)$ which is equal to $\mathcal{X}_\perp(A)$. ■

We are now ready to prove the conservation theorem (CT) for each of M^3 , ML^3 and QGL .

THEOREM 6.4. (CT for M^3 , ML^3 and QGL) *If Γ and A are classical then $\Gamma, \Box\Gamma \vdash_X \Box A$ implies that we have a classical proof of $\Gamma \vdash_X A$, where $X \in \{M^3, ML^3, QGL\}$.*

NOTE. This differs from 6.1 in that not only we have a proof for A from Γ alone, but the proof is classical. The key for this is that A itself is classical.

PROOF. Using 6.1 we obtain a proof of $\Gamma, \Box\Gamma \vdash_X A$, and using Lemma 6.3 we get a classical proof of $\mathcal{X}_\perp(\Gamma), \mathcal{X}_\perp(\Box\Gamma) \vdash_X \mathcal{X}_\perp(A)$. Since Γ and A are classical, we get a classical proof of $\Gamma, \top \vdash_X A$, that is, $\Gamma \vdash_X A$. ■

7. Strong Reflection and Strong Necessitation

Let us now give a proof of the fact that neither $\Box p \rightarrow p$ (strong reflection) nor $p \rightarrow \Box p$ (strong necessitation) are provable in either of the logics M^3 , ML^3 or QGL . Here p is a propositional variable (prime formula). This proof – unlike the one in [11] – does not rely on either cut elimination or “Gentzen-isation” of our logics, but rather utilises the formulator method developed here. *In fact, as already noted, QGL admits no cut elimination ([1]).*

REMARK 7.1. For every propositional variable, p , neither p nor $p \rightarrow \perp$ are provable in any of M^3 , ML^3 or QGL (which implies that they are consistent).

Indeed, assume that $\vdash p$, then by 6.3 (since p is classical), we could prove p classically; but a classical proof of p in these three logics is simply a proof of p in first order classical predicate logic (without equality) and we know, by soundness, that no such proof exists. Similar comment for $p \rightarrow \perp$. ■

THEOREM 7.2. *Neither $\Box p \rightarrow p$ (strong reflection) nor $p \rightarrow \Box p$ (strong necessitation) are provable in any of M^3 , ML^3 or QGL .*

PROOF. We write “ \vdash ” for provability in any of these three logics.

1. Assume that $\vdash \Box p \rightarrow p$. Then, by 5.5, we have $\vdash \mathcal{X}_\perp(\Box p \rightarrow p)$ or $\vdash \top \rightarrow p$, or simply, $\vdash p$, which is impossible by the previous remark.

2. Assume that $\vdash p \rightarrow \Box p$. First, if $\vdash \Box p$, then, by the conservation theorem, 6.4, $\vdash p$ which we know is false. Therefore, $\nvdash \Box p$ and thus, $\mathcal{X}_\top(\Box p) = \perp$. Now, by 5.5, we have $\vdash \mathcal{X}_\top(p \rightarrow \Box p)$ or $\vdash p \rightarrow \perp$, which is impossible by the previous remark. ■

8. The Failure of Löb's Axiom in M^3

We conclude with another negative result that we obtain with the formulator method. Let us introduce the \forall -formulator, \mathcal{F}_\forall .

DEFINITION 8.1. We let \mathfrak{F}_\forall be the formulator that, for every formula A , satisfies $\mathfrak{F}_\forall(\Box A) = \forall \mathfrak{F}_\forall(A)$, that is, it replaces all occurrences of \Box by \forall . ■

LEMMA 8.2. \mathfrak{F}_\forall is conservative for M^3 .

PROOF. By Proposition 4.8, we only need to show that $\vdash_{M^3} \mathfrak{F}_\forall(A)$, for all $A \in \Omega_{M^3}$. In this proof we abbreviate $\mathfrak{F}_\forall(A)$ by A' to improve readability.

1. $A = \Box(B \rightarrow C) \rightarrow \Box B \rightarrow \Box C$. Then $\mathfrak{F}_\forall(A) = \forall(B' \rightarrow C') \rightarrow \forall B' \rightarrow \forall C'$, which is derivable in M^3 .
2. $A = \Box B \rightarrow \Box \Box B$. Then $\mathfrak{F}_\forall(A) = \forall B' \rightarrow \forall \forall B'$, which is derivable in M^3 .
3. $A = \Box B \rightarrow \Box \forall x B$. Then $\mathfrak{F}_\forall(A) = \forall B' \rightarrow \forall \forall x B'$, which is derivable in M^3 .
4. $A \in \Box \Lambda_{M^3}$. Then $A = \Box B$ for some $B \in \Lambda_{M^3}$. Now, $\mathfrak{F}_\forall(A) = \forall B'$, which is provable in M^3 through generalisation(s), since B' is provable by 1–3 above, and by 4.5 for the classical axioms. ■

We can now state and prove the following theorem:

THEOREM 8.3. $\Box(\Box p \rightarrow p) \rightarrow \Box p$ is not provable in M^3 .

PROOF. Assume that $\vdash_{M^3} \Box(\Box p \rightarrow p) \rightarrow \Box p$, then, by 8.2, $\vdash_{M^3} \mathfrak{F}_\forall(\Box(\Box p \rightarrow p) \rightarrow \Box p)$ or $\vdash_{M^3} \forall(\forall p \rightarrow p) \rightarrow \forall p$ which is the same as $\vdash_{M^3} p$ which is impossible by 7.1. ■

References

- [1] A. Avron, *On modal systems having arithmetical interpretations*, **J. of Symb. Logic** 49, no.3 (1984), pp. 935–942.
- [2] George Boolos, **The unprovability of Consistency: An essay in modal logic**, Cambridge University Press, Cambridge, 1979.
- [3] N. Bourbaki, **Éléments de Mathématique: Théorie des Ensembles**, Hermann, Paris, 1966.
- [4] Herbert B. Enderton, **A mathematical introduction to logic**, Academic Press, New York, 1972.
- [5] F. Kibedi and G. Tourlakis, *A modal extension of weak generalisation predicate logic*, **Logic Journal of the IGPL** 14, no. 4 (2006), pp. 591–621.
- [6] Daniel Leivant, *On the proof theory of the modal logic for arithmetic provability*, **Journal of Symbolic Logic** 46, no. 3 (1981), pp. 531–538.
- [7] Elliott Mendelson, **Introduction to mathematical logic**, 3rd edition, Wadsworth & Brooks, Monterey, California, 1987.
- [8] G. Sambin and S. Valentini, *A Modal Sequent Calculus for a Fragment of Arithmetic*, **Studia Logica** 39, no. 2/3 (1980), pp. 245–256.
- [9] G. Sambin and S. Valentini, *The modal Logic of Provability. The Sequential Approach*, **Journal of Philosophical Logic** 11, no. 3 (1982), pp. 311–342.
- [10] K. Schütte, **Proof Theory**, Springer-Verlag, New York, 1977.
- [11] Y. Schwartz and G. Tourlakis, *On the proof-theory of two formalisations of modal first-order logic*, **Studia Logica** 96, no. 3 (2010), pp. 349–373.
- [12] Yehuda Schwartz, **Some Results in Computability and the Proof Theory of Predicate Modal Logic**, Ph.D. thesis, York University, Toronto, Canada, 2012.
- [13] C. Smoryński, **Self-Reference and Modal Logic**, Springer-Verlag, New York, 1985.
- [14] G. Tourlakis and F. Kibedi, *A modal extension of first order classical logic – Part I*, **Bulletin of the Section of Logic** 32/4 (2003), pp. 165–178.
- [15] G. Tourlakis and F. Kibedi, *A modal extension of first order classical logic – Part II*, **Bulletin of the Section of Logic** 33/1 (2004), pp. 1–10.
- [16] S. Valentini, *The Modal Logic of Provability: Cut-Elimination*, **Journal of Philosophical Logic** 12, no. 4 (1983), pp. 471–476.