Bulletin of the **S**ection of **L**ogic
Volume 42:3/4 (2013), pp. 119–134

Verónica Quiroga

AN ALTERNATIVE DEFINITION OF F-STRUCTURES FOR THE LOGIC C_1

Abstract

In this work we give a characterization of the F-structures defined by M. M. Fidel, which are the basis of the algebraic-relational semantics for the paraconsistent logic C_1. The new formulation is simpler and it allows a deeper study of such structures, as we will show in some applications in this article.

1. Introduction and Preliminaries

In his demonstration of the decidability of the paraconsistent Da Costa's logics C_n $(0 \leq n \leq \omega)$, M. M. Fidel in [7] introduced certain algebraic-relational structures, known nowadays as F-structures.

In particular, F-structures associated to the logic C_ω (or F_ω-structures, from now on) consist of systems of the form $\langle L, \{N_x\}_{x\in L}\rangle$, where L is a pseudo-complemented lattice,which associates a subset N_x to every $x \in L$. The aim of this construction is to interpret the paraconsistent nature of the hierarchy C_n, understanding that each element of the lattice has more than a negation. On the other hand, in the case of the logics C_n (with $n \in \mathbb{N}$), its associated F-structures (called as F_n-structures throughout this article) are enriched with a family $\{N_x^{(n)}\}_{x\in L}$. This new family interpret, in the algebraic context of L, the concept of "good-behavior" (w.r.t negation connectives) of the formulas, being this notion intrinsic to the definition of da Costa's logics.

In this way, every F_n-structure can be understood as an algebraic-relational system of the form $\langle L, \{N_x\}_{x\in L}, \{N_x^{(n)}\}_{x\in L}\rangle$, since every set N_x

and $N_x^{(n)}$ can be considered as a 1-ary predicate, as suggested in [12]. Despite the original purpose of M.M. Fidel in its definition, that we will recall in a formal way later, no significant studies of F-structures were done. This is because of two main reasons: the ulterior definition of simpler semantics of da Costa's logics, such as quasi-matrix semantics. Besides, the proof given in [10] that the logic C_1 is *not algebraizable* (results that can be generalized to every logic C_n) discouraged the study of the F-structures in depth (and, even, some algebraic works prior to Fidel's result were not taken into account).

However, given its conceptual richness, F-structures were adapted to other logics. For instance, in [9], the semantic of the F-structures for explosive Nelson's logic **N3** was developed by M.M. Fidel, while in [12], it was shown a generalization of these results for the paraconsistent logic **N4**. This suggests that this kind of structures deserves a deeper study. Now, turning back to the particular case of the C_n-logics, we arrive to a (new) disadvantage, related to this kind of models: in the case of C_n (with $n \in \mathbb{N}$) its definition is too complicated to allow a reasonable algebraic analysis [1]. This fact is probably caused by the original Fidel's notation, of course, but mainly because of the definition of the families $\{N_x\}_{x\in L}$ and $\{N_x^{(n)}\}_{x\in L}$ that are used in the C_n-logics.

This article shows an improvement of the original Fidel's definition, for the particular case of the logic C_1. Actually, the main result that will be presented here establishes that the relations $N_x^{(1)}$ can be defined directly from the sets N_x. Besides that, we will show in which way this new formulation of the F_1-structures can be applied to some examples.

To arrive to the desired result, this work is structured as follows: the next section will indicate the original axiomatics of C_ω and the family C_n (with $n \in \mathbb{N}$), together with its respective F-structures, as originally presented in [7]. In Section 3, the F_ω-structures are presented under a new formalism. This give us, in a simple way, the new characterization of F_1-structures, which will result equivalent to the original Fidel's definition. Finally, we show in the last section some examples of such structures, and some applications of the previous results.

[1] On the other hand, F-structures for the logic C_ω are relatively simple, and some researches about its related F-structures were already presented. See [13].

2. F_ω and F_1-structures

Just for a self-contained approach to our main results, we will introduce the axiomatics of the logics C_ω and C_n (with $n \in \mathbb{N}$), as presented in [4]. Even when those axiomatics were slightly simplified later, we chose them to sustain the original set of axioms because F-structures are defined according to the former version.

All the C_n-logics have the same set of connectives: $\{\neg, \vee, \wedge, \rightarrow\}$, with obvious meaning, generating a propositional language (indicated by $\mathcal{L}$ here) as usual. Besides, the basis of all the hierarchy C_n is the logic C_ω, which is axiomatized in a Hilbert-style in the following way:

Definition 2.1. The **logic $\boldsymbol{C_\omega}$** is given by means of a Hilbert-style axiomatic, with the following axiom schemas (where A, B, C are varying over formulas):

$A1)$	$A \rightarrow (B \rightarrow A)$	$A6)$	$A \rightarrow (B \rightarrow A \wedge B)$
$A2)$	$(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$	$A7)$	$A \rightarrow A \vee B$
$A3)$	$(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$	$A8)$	$B \rightarrow A \vee B$
$A4)$	$A \wedge B \rightarrow A$	$A9)$	$A \vee \neg A$
$A5)$	$A \wedge B \rightarrow B$	$A10)$	$\neg\neg A \rightarrow A$.

The only rule of inference used here is Modus Ponens: $\dfrac{A \rightarrow B,\ A}{B}$.

Consider now the following abbreviations:

$$\begin{aligned} A^o &:= \neg (A \& \neg A) \\ A^{(1)} &:= A^o \\ A^{(n)} &:= A^{(n-1)} \wedge \left(A^{(n-1)}\right)^o \end{aligned}$$

With these conventions in mind, it is defined:

Definition 2.2. For every $n \in \mathbb{N}$, the logic C_n is given by the axioms of C_ω, plus the following schema:

$A11)$ $B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$,

$A12)$ $A^{(n)} \,\&\, B^{(n)} \supset (A \,\&\, B)^{(n)}$,

$A13)$ $A^{(n)} \,\&\, B^{(n)} \supset (A \vee B)^{(n)}$,

$A14)$ $A^{(n)} \,\&\, B^{(n)} \supset (A \supset B)^{(n)}$.

As it is already known, the symbol $A^{(n)}$ indicates that A is "well-behaved" (w.r.t. non-contradiction principle). This is one of the distinguished characteristics of C_n: if A is a well-behaved formula, then the properties of classical logic are valid for it. With the precedent axiomatics in mind, we will define F_ω-structures and F_n-structures. First, recall these well-known definitions:

DEFINITION 2.3. An algebra $\langle L, \vee, \wedge, \Rightarrow, 1\rangle$ of type $(2, 2, 2, 0)$ is a relatively pseudocomplemented lattice (or R.P.L., for short) iff $\langle L, \vee, \wedge\rangle$ is a lattice and for every a, $b \in L$, there exists $sup\{x : x \wedge a \leq b\}$ and $a \Rightarrow b := sup\{x : x \wedge a \leq b\}$. The element $a \Rightarrow b$ is the **pseudocomplement of a relative to b**. In every R.P.L., its greatest element 1 is defined as $1{:=}x \Rightarrow x$ for any $x \in L$. On the other hand, a R.P.L. $\langle L, \vee, \wedge, \Rightarrow\rangle$ is a **Boolean algebra** (B.A.) iff it satisfies additionally $(a \Rightarrow b) \Rightarrow a \leq a$ (Pierce's Law) and has the zero element 0.

REMARK 2.4. Recall that every R.P.L. with 0 is a Heyting algebra (which in the context of the F_n-structures studied in this paper is not relevant).

Strictly speaking, the algebras of the form $\langle L, \vee, \wedge, \Rightarrow, 1\rangle$ are not Boolean algebras, but *bounded implicative classical lattices* (B.I.C.L.), since both structures have different similarity type. That is, if we define a B.A. as an algebra $\langle L, \vee, \wedge, \neg, 1, 0\rangle$ of type $(2, 2, 1, 0, 0)$, where $\langle L, \vee, \wedge\rangle$ is a distributive lattice verifying additionally: (i) $(a \wedge \neg a) \vee b = b$; (ii) $(a \vee \neg a) \wedge b = b$, then a B.A. is not a R.P.L. (because of the presence of $\neg$ instead of $\Rightarrow$). However, it is well known that both notions are interdefinable, where the relation between these operations is given by: $a \Rightarrow b = \neg a \vee b$ and $\neg a = a \Rightarrow 0$. See [3] for a proof of it. We have chosen to define a Boolean algebra in this way to respect the original spirit of Fidel's work.

DEFINITION 2.5. An $\boldsymbol{F_\omega}$**-structure** is a system $\langle L, \wedge, \vee, \Rightarrow, 1, \{N_x\}_{x \in L}\rangle$ such that:

(a) $\langle L, \wedge, \vee, \Rightarrow, 1\rangle$ is a R.P.L., 1 being the greatest element.

(b) $\{N_x\}_{x \in L}$ is a family of non-void subsets of L, verifying for each $x \in L$:

(c) If $x' \in N_x$, then $x \vee x' = 1$.

(d) For every $x' \in N_x$ exists $x'' \in N_{x'}$, such that $x'' \leq x$.

The underlying idea behind the sets $\{N_x\}_{x\in L}$, mentioned in [8], is that every $x' \in N_x$ is *a negation of* x (and not merely "the negation" of x). That is, in F_ω-structures, the elements $x \in L$ can have more than one negation. This property suggests the "paraconsistent character" of such structures.

The definition of F_n-structures takes in account the "well-behaved formulas" in the same way that the negations for F_ω-structures. That is, for every $x \in L$ is associated *a family* $N_x^{(n)} \subseteq L$, which can be understood as a family of "forms of good behavior of x". Formally:

Definition 2.6. An ***F*-structure for C_n** (or, simply, an F_n-structure) is a system $\left\langle L, \vee, \wedge, \Rightarrow, 1, 0, \{N_x\}_{x\in L}, \{N_x^{(n)}\}_{x\in L}\right\rangle$ such that:

(F-1) $\langle L, \wedge, \vee, \Rightarrow, 1, \{N_x\}_{x\in L}\rangle$ is an F_ω-structure and $\langle L, \wedge, \vee, \neg, 1, 0\rangle$ is a Boolean algebra, where $\neg x = x \Rightarrow 0$, for each $x \in L$.

(F-2) $\{N_x^{(n)}\}_{x\in L}$ is a family of non-void subsets of L.

(F-3) If $x' \in N_x$ and $y' \in N_y$, then exists $(x \wedge y)' \in N_{x\wedge y}$ such that $(x \wedge y)' \leq x' \vee y'$.

(F-4) If $x^{(n)} \in N_x^{(n)}$ and $y^{(n)} \in N_y^{(n)}$, then exist $(x \vee y)^{(n)}$ in $N_{x\vee y}^{(n)}$ and $(x \Rightarrow y)^{(n)}$ in $N_{(x\Rightarrow y)}^{(n)}$, such that $x^{(n)} \wedge y^{(n)} \leq (x \vee y)^{(n)}$ and $x^{(n)} \wedge y^{(n)} \leq (x \Rightarrow y)^{(n)}$.

(F-5) For every $x^{(n)} \in N_x^{(n)}$ exist $x' \in N_x$, $x'' \in N_{x'}$, $x^1 \in N_{x\wedge x'}$, $(x^1)' \in N_{x^1}$, $(x^1)'' \in N_{(x^1)'}$, $x^2 \in N_{x^1\wedge(x^1)'}$, $(x^2)' \in N_{x^2}$, $(x^2)'' \in N_{(x^2)'}$,..., $x^n \in N_{x^{n-1}\wedge(x^{n-1})'}$, such that:

   (a) $(x^k)'' \leq x^k$ (with $k = 0, \ldots, n-1$; $x^0 = x$);
   (b) $x^k \leq (x^{k-1})' \vee (x^{k-1})''$ (with $k = 1, \ldots, n$);
   (c) $(x^k)' \leq x^{k-1} \wedge (x^{k-1})'$ (with $k = 1, \ldots, n-1$);
   (d) $x^{(n)} = x^1 \wedge x^2 \wedge \cdots \wedge x^n$;
   (e) $x \wedge x' \wedge x^{(n)} = 0$.

(F-6) For every $x' \in N_x$, there are $x^{(n)} \in N_x^{(n)}$, $x'' \in N_{x'}$, $x^1 \in N_{x\wedge x'}$, $(x^1)' \in N_{x^1}$, $(x^1)'' \in N_{(x^1)'}$, $x^2 \in N_{x^1\wedge(x^1)'}$,..., $x^n \in N_{x^{n-1}\wedge(x^{n-1})'}$, such that the conditions (a)-(e) are satisfied.

Clearly, condition (F-4) is given to describe axioms $A13$) and $A14$) of C_n. On the other hand, axioms $A11$) and $A12$) are interpreted by means of conditions (F-5) and (F-6).

We conclude this section indicating that the class of F_ω-structures and F_1-structures define consequence relations $\models_{F_\omega}$ and $\models_{F_1}$, respectively, by means of convenient valuations (see [7]).

Definition 2.7. An **ω-valuation** is any mapping v of the set of formulas of C_ω into an F_ω-structure L, defined by induction on the length of a formula as follows:

(a) $v(P) \in L$, where P is a propositional variable;

If A and B are propositional variables or are of one of the forms: $C \wedge D$, $C \vee D$ or $C \to D$, then

(b) $v(A \wedge B) = v(A) \wedge v(B)$, $v(A \vee B) = v(A) \vee v(B)$, $v(A \to B) = v(A) \to v(B)$;

(c) $v(\neg A) \in N_{v(A)}$;

(d) $v(\neg\neg A) \in N_{v(\neg A)}$ and $v(\neg\neg A) \leq v(\neg A)$.

An **1-valuation** is any mapping v of the set of formulas of C_n into an F_1-structure L, defined by induction on the length of a formula as follows:

(a) $v(P) \in L$, $v\left(P^{(1)}\right) \in N^{(1)}_{v(P)}$, $v(\neg P) \in N_{v(P)}$, $v(\neg\neg P) \in N_{v(\neg P)}$, $v\left(P^{(1)}\right) \in N_{v(P\wedge\neg P)}$, $v\left(\neg P^{(1)}\right) \in N_{v\left(P^{(1)}\right)}$, $v\left(\neg\neg P^{(1)}\right) \in N_{v\left(\neg P^{(1)}\right)}$, where P is a propositional variable and the following conditions are satisfied: $v(P \wedge \neg P) = v(P) \wedge v(\neg P)$; $v(\neg\neg P) \leq v(P)$; $v\left(P^{(1)}\right) \leq v(\neg P) \vee v(\neg\neg P)$; $v\left(\neg P^{(1)}\right) \leq v(P) \wedge v(\neg P)$; $v(P) \wedge v(\neg P) \wedge v(P^{(1)}) = 0$.

(b) If A and B are propositional variables or are of one of the forms: $C \wedge D$, $C \vee D$ or $C \to D$, then $v(A \wedge B) = v(A) \wedge v(B)$, $v(A \vee B) = v(A) \vee v(B)$, $v(A \to B) = v(A) \to v(B)$;

(c) If A and B are propositional variables or are of one of the forms: $C \wedge D$, $C \vee D$ or $C \to D$, then $v\left((A \vee B)^{(1)}\right) \in N^{(1)}_{v(A\vee B)}$, $v(A^{(1)}) \wedge v(B^{(1)}) \leq v\left((A \vee B)^{(1)}\right)$, $v\left((A \to B)^{(1)}\right) \in N^{(1)}_{v(A\to B)}$, $v(A^{(1)}) \wedge v(B^{(1)}) \leq v\left((A \to B)^{(1)}\right)$.

(d) If A is $B \vee C$ or $B \to C$, given $v(A^{(1)}) \in N^{(1)}_{v(A)}$, we take $v(\neg A) \in N_{v(A)}$, $v(\neg\neg A) \in N_{v(\neg A)}$, $v(A^{(1)}) \in N_{v(A\wedge\neg A)}$, $v\left(\neg A^{(1)}\right) \in N_{v\left(A^{(1)}\right)}$, $v\left(\neg\neg A^{(1)}\right) \in N_{v\left(\neg A^{(1)}\right)}$, such that:

(i) $v(\neg\neg A) \leq v(A)$;

(ii) $v\left(A^{(1)}\right) \leq v\left(\neg A\right) \vee v\left(\neg\neg A\right);$
(iii) $v\left(\neg A^{(1)}\right) \leq v\left(A\right) \wedge v\left(\neg A\right);$
(iv) $v(A) \wedge v(\neg A) \wedge v(A^{(1)}) = 0.$

(e) If A is $\neg B$ or $B \wedge C$, given $v(\neg A) \in N_{v(A)}$, we choose $v(A^{(1)}) \in N^{(1)}_{v(A)}$, $v(\neg\neg A) \in N_{v(\neg A)}$, $v(A^{(1)}) \in N_{v(A\wedge\neg A)}$, $v\left(\neg A^{(1)}\right) \in N_{v\left(A^{(1)}\right)}$, $v\left(\neg\neg A^{(1)}\right) \in N_{v\left(\neg A^{(1)}\right)}$, such that the conditions (i)-(iv) of (d) are verified.

The ω-valuations and 1-valuations are not in general determined by their values on the set of propositional variables.

DEFINITION 2.8. The formula A is called **$\boldsymbol{\omega}$-valid** (**1-valid**) in an F_ω (F_1)-structure L, and we shall write $\models^L_\omega A$ $\left(\models^L_1 A\right)$, if $v(A) = 1$ for any ω-valuation (1-valuation) v into L. A is valid, and we shall write $\models_\omega A$ $(\models_1 A)$, if for every F_ω (F_1)-structure L, it is true that $\models_\omega A$ $(\models_1 A)$.

THEOREM 2.9. *For every formula $A \in \mathcal{L}$, are verified:*

(a) $\vdash_{C_\omega} A$ *iff* $\models_{F_\omega} A$
(b) $\vdash_{C_1} A$ *iff* $\models_{F_1} A$

3. A new characterization of F_1-structures

It is obvious that the applied formalism can be modified having in mind better applications of F_ω-structures. First of all, the family $\{N_x\}_{x\in L}$ can be obviously formalized as a function $\mathfrak{f} : L \to \wp(L)$. Besides that, recall the notion of annihilator of a relative to b, indebted to M. Mandelker (see [11]): $\langle a, b\rangle^\circ{:=}\{x \in L : x \wedge a \leq b\}$, and its particular case $\langle a, 0\rangle^\circ$ (denoted simply by a°). Even when annihilators were deeply studied (see [1], [2] or [5]), its dual notion was not very used. It will be useful for the definition of F_ω-structures.

DEFINITION 3.1. Let $\langle L, \vee, \wedge, 1\rangle$ be a lattice with greatest element 1. For every a, $b \in L$, the **para-annihilator of $\boldsymbol{a}$ relative to $\boldsymbol{b}$**, denoted by $\langle a, b\rangle^\intercal$, is the set $\langle a, b\rangle^\intercal := \{x \in L : x \vee a \geq b\}$. In particular, the **para-annihilator of $\boldsymbol{a}$** is simply $\langle a, 1\rangle^\intercal = \{x \in L : x \vee a = 1\}$. This set will be indicated as $a^\intercal$.

The following two propositions refer to the $F_{\boldsymbol{\omega}}$-structures. The proofs are immediate from Definitions 2.5 and 3.1.

Proposition 3.2. *Let $\langle L, \vee, \wedge, \Rightarrow, 1, \{N_x\}_{x\in L}\rangle$ be an $F_{\boldsymbol{\omega}}$-structure. Then there is a function $\mathfrak{f} : L \to \wp(L)$ defined by $\mathfrak{f}(x) = N_x$ such that for any $x \in L$ satisfies:*

(a) $\emptyset \subset \mathfrak{f}(x) \subseteq x^{\mathsf{T}}$*;*
(b) $\mathfrak{f}(y) \cap \downarrow x \neq \emptyset$*, for every $y \in f(x)$.*

Of course, here $\downarrow x$ ($\uparrow x$) denotes the down-set (up-set) relative to x, as usual (see [6], for example).

Proposition 3.3. *Let $\langle L, \vee, \wedge, \Rightarrow, 1\rangle$ be a R.P.L. If $\mathfrak{f} : L \to \wp(L)$ is a function that verifies the conditions* (a) *and* (b) *of the Proposition 3.2, then $\langle L, \vee, \wedge, \Rightarrow, 1, \{\mathfrak{f}(x)\}_{x\in L}\rangle$ is an $F_{\boldsymbol{\omega}}$-structure.*

From Propositions 3.2 and 3.3, we can characterize the $F_{\boldsymbol{\omega}}$-structures by means of a function $\mathfrak{f}$ that satisfies certain conditions. In what follows, we use the notation $\langle L, \vee, \wedge, \Rightarrow, 1, \mathfrak{f}\rangle$ to refer to the 6-tuple $\langle L, \vee, \wedge, \Rightarrow, 1, \{N_x\}_{x\in L}\rangle$.

Similarly, we can consider formulations concerning the F_1-structures using two functions. The proofs are immediate from Definition 2.6 considering $n = 1$.

Proposition 3.4. *Let $\left\langle L, \vee, \wedge, \Rightarrow, 1, 0, \{N_x\}_{x\in L}, \left\{N_x^{(1)}\right\}_{x\in L}\right\rangle$ be an F_1-structure. Then there are two functions $\mathfrak{f} : L \to \wp(L)$ and $\mathfrak{F} : L \to \wp(L)$ defined by $\mathfrak{f}(x) = N_x$ and $\mathfrak{F}(x) = N_x^{(1)}$, respectively, such that:*

(f-1) *$\langle L, \vee, \wedge, \Rightarrow, 1, \mathfrak{f}\rangle$ is an F_ω-structure and $\langle L, \vee, \wedge, \neg, 1, 0\rangle$ is a Boolean algebra, where $\neg x = x \Rightarrow 0$, for each $x \in L$*
(f-2) *The function $\mathfrak{F} : L \longrightarrow \wp(L)$ verifies $\mathfrak{F}(x) \neq \emptyset$, for every $x \in L$.*
(f-3) *For every $x,\ y \in L$, for every $z \in \mathfrak{f}(x)$ and $w \in \mathfrak{f}(y)$, it holds that $\mathfrak{f}(x \wedge y) \cap \downarrow (z \vee w) \neq \emptyset$.*
(f-4) *For every $x,\ y \in L$, for every $z \in \mathfrak{F}(x)$, for every $w \in \mathfrak{F}(y)$, it is satisfied:*
  (a) $\mathfrak{F}(x \vee y) \cap \uparrow (z \wedge w) \neq \emptyset$.
  (b) $\mathfrak{F}(x \Rightarrow y) \cap \uparrow (z \wedge w) \neq \emptyset$.

(f-5) *For every $x \in L$ and $z \in \mathfrak{F}(x)$, there are $y \in \mathfrak{f}(x)$, $u \in \mathfrak{f}(y)$ and $v \in \mathfrak{f}(z)$, verifying:*

(a) $u \leq x$;
(b) $z \leq y \vee u$;
(c) $v \leq x \wedge y$;
(d) $z \in f(x \wedge y)$;
(e) $x \wedge y \wedge z = 0$.

(f-6) *For every $x \in L$, for every $y \in \mathfrak{f}(x)$, there are $z \in \mathfrak{F}(x)$, $u \in \mathfrak{f}(y)$, $v \in \mathfrak{f}(z)$, such that the conditions* (a)-(e) are satisfied.

PROPOSITION 3.5. *Let $\langle L, \vee, \wedge, \neg, 1, 0\rangle$ is a Boolean algebra. If $\mathfrak{f} : L \to \wp(L)$ and $\mathfrak{F} : L \to \wp(L)$ are two functions that verify* (f-1)-(f-6) of Proposition 3.4, then $\left\langle L, \vee, \wedge, \Rightarrow, 1, 0, \{\mathfrak{f}(x)\}_{x\in L}, \{\mathfrak{F}(x)\}_{x\in L}\right\rangle$ is an F_1-structure, where $x \Rightarrow y = \neg x \vee y$, for every $x, y \in L$.

From now on, according to Propositions 3.4 and 3.5, we can write $\langle L, \vee, \wedge, \Rightarrow, 1, 0, \mathfrak{f}, \mathfrak{F}\rangle$ instead of $\left\langle L, \vee, \wedge, \Rightarrow, 1, 0, \{N_x\}_{x\in L}, \left\{N_x^{(1)}\right\}_{x\in L}\right\rangle$. The main result of this paper is to establish a simple and practical characterization of an F_1-structure. This is possible because $\mathfrak{F}$ can be defined by means of $\mathfrak{f}$.

DEFINITION 3.6. An $\overline{\boldsymbol{F}}_{\mathbf{1}}$**-structure** is a system $\langle L, \vee, \wedge, \neg, 1, 0, \mathfrak{f}\rangle$ such that

(F_1-1) $\langle L, \vee, \wedge, \neg, 1, 0\rangle$ is a Boolean algebra.

(F_1-2) $\mathfrak{f} : L \to \wp(L)$ is a function that verifies $\neg x \in \mathfrak{f}(x) \subseteq x^{\intercal}$, for every $x \in L$.

THEOREM 3.7. *If $\langle L, \vee, \wedge, \neg, 1, 0, \mathfrak{f}\rangle$ is an $\overline{F}_1$-structure, then $\langle L, \vee, \wedge, \Rightarrow, 1, 0, \mathfrak{f}, \mathfrak{F}\rangle$ is an F_1-structure, where $x \Rightarrow y = \neg x \vee y$, for every $x, y \in L$, and $\mathfrak{F}(x) := \{\neg x \vee \neg a : a \in \mathfrak{f}(x)\}$, for every $x \in L$.*

THEOREM 3.8. *If $\langle L, \vee, \wedge, \Rightarrow, 1, 0, \mathfrak{f}, \mathfrak{F}\rangle$ is an F_1-structure, then $\langle L, \vee, \wedge, \neg, 1, 0, \mathfrak{f}\rangle$ is an $\overline{F}_1$-structure, where $\neg x = x \Rightarrow 0$, for each $x \in L$. In addition, $\mathfrak{F}(x) = \{\neg x \vee \neg a : a \in \mathfrak{f}(x)\}$, for every $x \in L$.*

To prove Theorems 3.7 and 3.8, we have to use some technical results. First, it is easy to see the following:

Proposition 3.9. *For every Boolean algebra A, for every $x \in A$, $\uparrow (\neg x) = x^{\intercal}$.*

Corollary 3.10. *If $\langle L, \vee, \wedge, \neg, 1, 0, \mathfrak{f}\rangle$ is an $\overline{F}_1$-structure, then $\mathfrak{f}(x) \subseteq \uparrow (\neg x)$.*

Proposition 3.11. *If $\langle L, \vee, \wedge, \neg, 1, 0, \mathfrak{f}\rangle$ is an $\overline{F}_1$-structure, then $\langle L, \vee, \wedge, \Rightarrow, 1, \mathfrak{f}\rangle$ is an F_ω-structure, where $x \Rightarrow y = \neg x \vee y$, for every $x, y \in L$.*

Proof: It is well known that if $\langle L, \vee, \wedge, \neg, 1, 0\rangle$ is a Boolean algebra and $x \Rightarrow y = \neg x \vee y$, for every $x, y \in L$, then $\langle L, \vee, \wedge, \Rightarrow, 1\rangle$ is a R.P.L. Furthermore, we consider $x \in L$, then by (F_1-2), $\emptyset \subset \mathfrak{f}(x) \subseteq x^{\intercal}$. Now, suppose $y \in \mathfrak{f}(x)$, according to the Corollary 3.10, $\neg x \leq y$ and, so, $\neg y \leq x$. And since $\neg y \in \mathfrak{f}(y)$ (because (F_1-2)), it follows that $\mathfrak{f}(y) \cap \downarrow x \neq \emptyset$. By Proposition 3.3, $\langle L, \vee, \wedge, \Rightarrow, 1, \mathfrak{f}\rangle$ is an F_ω-structure. □

Proposition 3.12. *If $\langle L, \vee, \wedge, \neg, 1, 0, \mathfrak{f}\rangle$ is an $\overline{F}_1$-structure and, for every $x \in L$, we define $\mathfrak{F}(x) := \{\neg x \vee \neg a : a \in \mathfrak{f}(x)\}$, then $1 \in \mathfrak{F}(x)$.*

Proof: By (F_1-2), for every $x \in L$, we have $\neg x \in \mathfrak{f}(x)$. So, $1 = \neg x \vee \neg\neg x \in \mathfrak{F}(x)$. □

Now, we give a proof of Theorem 3.7.

Proof of Theorem 3.7: Suppose $\langle L, \vee, \wedge, \neg, 1, 0, \mathfrak{f}\rangle$ is an $\overline{F}_1$-structure. Also, we define $\mathfrak{F} : L \to \wp(L)$ by $\mathfrak{F}(x) = \{\neg x \vee \neg a : a \in \mathfrak{f}(x)\}$. Let us prove properties (f-1)−(f-6) indicated in Proposition 3.4. By Proposition 3.11, (f-1) is satisfied. Besides, Proposition 3.12 proves (f-2).

To prove (f-3), consider x, $y \in L$ and $z \in \mathfrak{f}(x)$, $w \in \mathfrak{f}(y)$. By Corollary 3.10, $\neg x \leq z$ and $\neg y \leq w$. So, $\neg(x \wedge y) \leq z \vee w$ and, by (F_1-2), $\neg(x \wedge y) \in \mathfrak{f}(x \wedge y)$. Therefore, (f-3) is also verified.

On the other hand, suppose x, $y \in L$, $z \in \mathfrak{F}(x)$, $w \in \mathfrak{F}(y)$. By Proposition 3.12, $1 \in \mathfrak{F}(x \vee y)$ and since $z \wedge w \leq 1$, we have that $\mathfrak{F}(x \vee y) \cap \uparrow (z \wedge w) \neq \emptyset$. Moreover, $x \Rightarrow y = \neg x \vee y$, thus $1 \in \mathfrak{F}(x \Rightarrow y) \cap \uparrow (z \wedge w)$. This demonstrates (f-4).

Let us prove (f-5) now: for that, consider $x \in L$ and $z \in \mathfrak{F}(x)$. By definition of $\mathfrak{F}$, there is $y \in \mathfrak{f}(x)$ such that $z = \neg x \vee \neg y = \neg(x \wedge y)$, which implies $z \in (x \wedge y)^o$. Besides, $z \in \mathfrak{f}(\neg z) = \mathfrak{f}(x \wedge y)$, and so $z \in \mathfrak{f}(x \wedge y) \cap (x \wedge y)^o$. Hence, y satisfies (f-5) (d) and (e). Now, we know that (f-1) is verified, that is, $\langle L, \vee, \wedge, \Rightarrow, 1, \mathfrak{f}\rangle$ is an F_ω-structure and since

$y \in \mathfrak{f}(x)$, exists $u \in \mathfrak{f}(y)$ such that $u \leq x$. Thus, (f-5)(a) is valid. Similarly, there is $v \in \mathfrak{f}(z)$ with $v \leq x \wedge y$, verifying (f-5) (c). Finally, $z \leq 1 = y \vee u$ (because $u \in \mathfrak{f}(y)$). That is, (f-5) (b) is verified.

For (f-6), let y be in $\mathfrak{f}(x)$. By definition of $\mathfrak{F}$, $z := \neg(x \wedge y) = \neg x \vee \neg y \in \mathfrak{F}(x)$ and by (F_1-2), following that $z \in \mathfrak{f}(x \wedge y)$. So, $z \in \mathfrak{F}(x) \cap \mathfrak{f}(x \wedge y) \cap (x \wedge y)^o$. As in the previous case, (f-6) (d) and (e) are verified. Moreover, there are $u \in \mathfrak{f}(y)$, $v \in \mathfrak{f}(z)$ validating (f-6) (a) and (c). And (f-6) (b) is valid because $y \vee u = 1$. Finally, by Proposition 3.5, $\langle L, \vee, \wedge, \Rightarrow, 1, 0, \mathfrak{f}, \mathfrak{F}\rangle$ is an F_1-structure. □

For the proof of Theorem 3.8, we consider the following facts:

PROPOSITION 3.13. *If $\langle L, \vee, \wedge, \Rightarrow, 1, 0, \mathfrak{f}, \mathfrak{F}\rangle$ is an F_1-structure and $\neg x = x \Rightarrow 0$, for every $x \in L$, then for every $x \in L$, $\mathfrak{F}(x) = \{\neg x \vee \neg a : a \in \mathfrak{f}(x)\}$.*

PROOF: If $m \in \mathfrak{F}(x)$ then, by (f-5) (d) and (e) indicated in Proposition 3.4, exists $a \in \mathfrak{f}(x)$ such that $m \in \mathfrak{f}(x \wedge a) \cap (x \wedge a)^o$. From this and (f-1), $m \vee (x \wedge a) = 1$ and $m \wedge (x \wedge a) = 0$. Thus, $m = \neg(x \wedge a) = \neg x \vee \neg a$. Therefore, $m \in \{\neg x \vee \neg a : a \in \mathfrak{f}(x)\}$. On the other hand, if $m \in \{\neg x \vee \neg a : a \in \mathfrak{f}(x)\}$, then $m = \neg(x \wedge a)$ for some $a \in \mathfrak{f}(x)$. By (f-6) (d) and (e), exists $z \in \mathfrak{F}(x)$ with $z \in \mathfrak{f}(x \wedge a)$ and $(x \wedge a) \wedge z = 0$. Again, $z \vee (x \wedge a) = 1$ because (f-1), and so $m = z$, which implies that $m \in \mathfrak{F}(x)$. □

PROPOSITION 3.14. *If $\langle L, \vee, \wedge, \Rightarrow, 1, 0, \mathfrak{f}, \mathfrak{F}\rangle$ is an F_1-structure and $\neg x = x \Rightarrow 0$, for every $x \in L$, then*

(a) *$\neg x \in \mathfrak{f}(x)$ iff $x \in \mathfrak{f}(\neg x)$.*
(b) *If $a \in \mathfrak{f}(x)$ for some $x \in L$, then $a \in \mathfrak{f}(\neg a)$ and $\neg a \in \mathfrak{f}(a)$.*

PROOF: If $\neg x \in \mathfrak{f}(x)$, then by (f-1) of Proposition 3.4, exists $a \leq x$ such that $a \in \mathfrak{f}(\neg x) \subseteq (\neg x)^{\intercal} = \uparrow x$ (see Proposition 3.9). Thus, $x = a$, which implies $x \in \mathfrak{f}(\neg x)$. This entails (a) because L is a Boolean algebra according to Proposition 3.4. For (b), suppose $a \in \mathfrak{f}(x)$ for some $x \in L$. It was already showed that, by (f-6) (d) and (e), exists $m \in \mathfrak{f}(x \wedge a) \cap (x \wedge a)^o$. So, $m \vee (x \wedge a) = 1$ and $m \wedge (x \wedge a) = 0$. Then $\neg(x \wedge a) = m \in \mathfrak{f}(x \wedge a)$ and consequently, by (a), $x \wedge a \in \mathfrak{f}(\neg(x \wedge a))$, too. So, applying (f-3), we get that exists t satisfying: $(*)$ $t \leq a \vee (x \wedge a) = a$ and $(**)$ $t \in \mathfrak{f}(x \wedge \neg(x \wedge a)) = \mathfrak{f}(x \wedge \neg a)$.

Now, since $a \in \mathfrak{f}(x) \subseteq x^{\intercal} = \uparrow(\neg x)$, we get $\neg x \leq a$, and so $x \wedge \neg a = \neg a$. Thus, by $(**)$, $t \in \mathfrak{f}(\neg a) \subseteq (\neg a)^{\intercal} = \uparrow a$. That is, $a \leq t$. So, by $(*)$, $a = t \in \mathfrak{f}(\neg a)$. And by (a), $\neg a \in \mathfrak{f}(a)$. This concludes the proof. □

Proposition 3.15. *If* $\langle L,\vee,\wedge,\Rightarrow,1,0,\mathfrak{f},\mathfrak{F}\rangle$ *is an* F_1*-structure and* $\neg x = x \Rightarrow 0$*, for every* $x \in L$*, then* $\langle L,\vee,\wedge,\neg,1,0,\mathfrak{f}\rangle$ *is an* $\overline{F}_1$*-structure.*

Proof: Let $\langle L,\vee,\wedge,\Rightarrow,1,0,\mathfrak{f},\mathfrak{F}\rangle$ be an F_1-structure. Hence, the conditions (f-1) −(f-6) of Proposition 3.4 are valid. So, $\langle L,\vee,\wedge,\Rightarrow,1,\mathfrak{f}\rangle$ is an F_ω-structure and $\langle L,\vee,\wedge,\neg,1,0\rangle$ is a Boolean algebra, where $\neg x = x \Rightarrow 0$, for each $x \in L$.

Now, we consider $x \in L$. By Proposition 3.2, $\mathfrak{f}(x) \subseteq x^{\intercal}$. Besides that, since $\mathfrak{F}(x) \neq \emptyset$, exists $p \in \mathfrak{f}(x)$ such that $\neg x \vee \neg p \in \mathfrak{F}(x)$ $(*)$ (see Proposition 3.13). Also, by Proposition 3.2, there is $y \in \mathfrak{f}(\neg x)$ and therefore $m \in \mathfrak{f}(y)$ such that $m \leq \neg x$. Applying Proposition 3.14 (b), we have $\neg m \in \mathfrak{f}(m)$ and by Proposition 3.13 again, $1 = \neg m \vee \neg\neg m \in \mathfrak{F}(m)$ $(**)$. So, by $(*)$, $(**)$ and Proposition 3.4 (f-4) (b), exists $t \in \mathfrak{F}(x \Rightarrow m) = \mathfrak{F}(\neg x \vee m) = \mathfrak{F}(\neg x)$ such that $t \geq (\neg x \vee \neg p) \wedge 1 = \neg x \vee \neg p$. This implies $\neg t \wedge (\neg x \vee \neg p) = 0$. Besides, since $t \in \mathfrak{F}(\neg x)$, by Proposition 3.13, $t = \neg\neg x \ \vee \neg t' = x \vee \neg t'$ (with $t' \in \mathfrak{f}(\neg x)$), and so $x \leq t$. Then, $\neg t \leq \neg x \leq \neg x \vee \neg p$. Hence, $\neg t = \neg t \wedge (\neg x \vee \neg p) = 0$.

On the other hand, using that $t = x \vee \neg t'$ with $t' \in \mathfrak{f}(\neg x)$ again, it follows that $0 = \neg t = \neg(x \vee \neg t') = \neg x \wedge t'$. In addition, $\neg x \vee t' = 1$ since L is an F_ω-structure. So, $x = \neg\neg x = t' \in \mathfrak{f}(\neg x)$. So, by Proposition 3.14 (a), the proof is completed. □

The proof of Theorem 3.8 is now complete by means of Propositions 3.15 and 3.13.

Corollary 3.16. *Let* $(L,\vee,\wedge,\neg,1,0)$ *be a Boolean algebra, and* $\mathfrak{f} : L \to \wp(L)$ *a function. Then,* $\langle L,\vee,\wedge,\Rightarrow,1,0,\mathfrak{f},\mathfrak{F}\rangle$ *is an* F_1*-structure (where* $x \Rightarrow y = \neg x \vee y$*, for every* $x, y \in L$*, and* $\mathfrak{F}(x) = \{\neg x \vee \neg a : a \in \mathfrak{f}(x)\}$*, for every* $x \in L$*) iff for every* $x \in L$*,* $\neg x$ *is the smallest element of* $\mathfrak{f}(x)$*.*

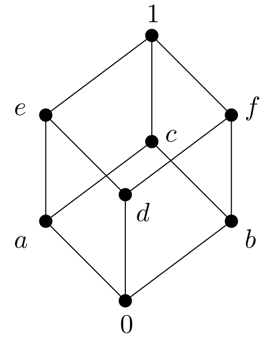
Proof:
($\Rightarrow$) From Theorem 3.8, (F_1-2) of Definition 3.6 and Corollary 3.10.
($\Leftarrow$) Suppose $\neg x$ is the smallest element of $\mathfrak{f}(x)$. Then, $y \in \mathfrak{f}(x)$ implies $\neg x \leq y$, and so $y \in x^{\intercal}$. Hence, $\mathfrak{f}(x) \subseteq x^{\intercal}$. Since $\neg x \in \mathfrak{f}(x)$, we have that $\langle L,\vee,\wedge,\neg,1,0,\mathfrak{f}\rangle$ is an $\overline{F}_1$-structure. By Theorem 3.7, $\langle L,\vee,\wedge,\Rightarrow,1,0,\mathfrak{f},\mathfrak{F}\rangle$ is an F_1-structure. □

Given that $\mathfrak{F}$ can be defined by means of $\mathfrak{f}$, from now on, we write $\langle L, \mathfrak{f}\rangle$ instead of $\langle L, \vee, \wedge, \Rightarrow, 1, 0, \mathfrak{f}, \mathfrak{F}\rangle$.

4. Applications and examples

In this section we show the simplicity of the analysis of F_1-structures using the new characterization.

EXAMPLE 4.1. Let L be the Boolean algebra with 3 atoms:



and let us consider the applications $\mathfrak{f}_1$ and $\mathfrak{f}_2$, as showed in the following table:

x	0	a	b	c	d	e	f	1
$\neg x$	1	f	e	d	c	b	a	0
$\mathfrak{f}_1(x)$	$\{1\}$	$\{f\}$	$\{e,1\}$	$\{d,f,1\}$	$\{c,1\}$	$\{b\}$	$\{a,c,1\}$	$\{0,a\}$
$\mathfrak{f}_2(x)$	$\{1\}$	$\{f\}$	$\{e,1\}$	$\{f,1\}$	$\{1\}$	$\{b\}$	$\{a,c,1\}$	$\{0,a\}$

By Definition 3.6 and Theorem 3.7, $\langle L, \mathfrak{f}_1\rangle$ is an F_1-structure. However, if we suppose that $\langle L, \mathfrak{f}_2\rangle$ is an F_1-structure, by Theorem 3.8, $\neg c = d \in \mathfrak{f}_2(c)$, but clearly this is not true. Therefore, our assumption is false.

By the way, this example shows that the reverse of Proposition 3.11 is not valid because it can be verified that $\langle L, \mathfrak{f}_2\rangle$ is an F_ω-structure, but (F_1-2) is not verified. □

THEOREM 4.2. *Let $\langle L, \mathfrak{f}\rangle$ be an F_1-structure. If $\equiv$ is a congruence defined on L, then $\langle L/\equiv, \bar{\mathfrak{f}}\rangle$ is an F_1-structure with $\bar{\mathfrak{f}}(\overline{x}) = \{\overline{y} : y \in \mathfrak{f}(x)\}$, where $\overline{x}$ denotes the equivalence class of x.*

Proof: Of course, $L/\equiv$ is a Boolean algebra. On the other hand, for every $x \in L$, $\neg x \in \mathfrak{f}(x) \subseteq x^{\intercal}$ and, consequently, $\neg\overline{x} \in \bar{\mathfrak{f}}(\overline{x})$. Now, take $\overline{y} \in \bar{\mathfrak{f}}(\overline{x})$. Then, by the definition of $\bar{\mathfrak{f}}$, it follows that $y \in \mathfrak{f}(x)$. Hence $\overline{y} \vee \overline{x} = \overline{y \vee x} = \overline{1}$ and, so $\overline{y} \in \overline{x}^{\intercal}$. By Definition 3.6 and Theorem 3.7, the proof is complete. □

Note that the previous result was demonstrated in [7], but our proof is simpler.

In the sequel we will work with F_1-structures having additional properties.

Proposition 4.3. *Let L be a Boolean algebra. If $\mathfrak{f} : L \to \wp(L)$ verifies $\mathfrak{f}(x) = x^{\intercal}$, for every $x \in L$, then $\langle L, \mathfrak{f} \rangle$ is an F_1-structure.*

Proof: It is immediate from Definition 3.6 and Theorem 3.7. □

Proposition 4.4. *If $\langle L, \mathfrak{f} \rangle$ is an F_1-structure such that $\mathfrak{f}(x) = x^{\intercal}$, for every $x \in L$, then for any $x \in L$, $\mathfrak{F}(x) = \{\neg x \vee \neg a : a \in \mathfrak{f}(x)\} = \mathfrak{f}(x)$.*

Proof: Let $\langle L, \mathfrak{f} \rangle$ be an F_1-structure such that $\mathfrak{f}(x) = x^{\intercal}$, for every $x \in L$. We consider $m \in \mathfrak{F}(x)$, then $m = \neg x \vee \neg a$, with $a \in \mathfrak{f}(x)$. Hence, $x \vee m = x \vee (\neg x \vee \neg a) = 1$, and so $m \in x^{\intercal} = \mathfrak{f}(x)$. On the other hand, if $m \in \mathfrak{f}(x)$, by Theorem 3.8 and Corollary 3.10, $\neg x \vee m = m$. In addition, since $(\neg x \vee \neg m) \vee x = 1$, we have that $\neg x \vee \neg m \in x^{\intercal} = \mathfrak{f}(x)$. Thus, $\neg x \vee \neg(\neg x \vee \neg m) \in \mathfrak{F}(x)$ and so $m \in \mathfrak{F}(x)$, because $\neg x \vee \neg(\neg x \vee \neg m) = m$. □

Definition 4.5. An **atomic F_1-structure** $\langle L, \mathfrak{f} \rangle$ is an F_1-structure such that L is an atomic Boolean algebra. In this context we denote the set $\{h : \neg h \text{ is an atom of } L\}$ by $\mathcal{A}$.

The next result arises from Definition 4.5 and Proposition 3.9.

Lemma 4.6. *Let $\langle L, \mathfrak{f} \rangle$ be an F_1-structure. Then, are valid:*

(a) *L is atomic iff for every $x \in L \setminus \{1\}$, $\mathcal{A} \cap \uparrow x \neq \emptyset$.*

(b) *If $\mathfrak{f}(x) = x^{\intercal}$, for every $x \in L$, then $\mathcal{A} \cap \uparrow x = \mathcal{A} \setminus \mathfrak{f}(x)$, for every $x \in L \setminus \{1\}$.*

THEOREM 4.7. *If $\langle L, \mathfrak{f}\rangle$ is an atomic F_1-structure such that $\mathfrak{f}(x) = x^{\intercal}$, for every $x \in L$, then $\mathfrak{f}(x) = \bigcap_{h\in\mathcal{A}\cap\uparrow x} \mathfrak{f}(h) = \bigcap_{h\in \mathcal{A}\setminus\mathfrak{f}(x)} \mathfrak{f}(h)$, for every $x \in L\setminus\{1\}$.*

PROOF: If $x \in L\setminus\{1\}$ then $\mathfrak{f}(x) = x^{\intercal} \subseteq h^{\intercal} = \mathfrak{f}(h)$, for every $h \in \mathcal{A}\cap\uparrow x$. Suppose now that $y \in \bigcap_{h\in\mathcal{A}\cap\uparrow x} \mathfrak{f}(h)$ but $x\vee y \neq 1$. By Lemma 4.6 (a), there is $k \in \mathcal{A}$ such that $x \vee y \leq k$, which implies $k \in\mathcal{A}\cap\uparrow x$ and, so, $y \in \mathfrak{f}(k)$. Note that $k = (x \vee y) \vee k = x \vee (y \vee k) = x \vee 1 = 1$ and, so, $k \notin \mathcal{A}$, which is absurd. Thus, $x \vee y = 1$. That is, $y \in \mathfrak{f}(x)$. The first equality is proved. On the other hand, by Lemma 4.6 (b), we obtain $f(x) = \bigcap_{h\in \mathcal{A}\setminus\mathfrak{f}(x)} f(h)$. □

EXAMPLE 4.8. Theorem 4.7 is not valid if the F_1-structures are not atomic. Consider as example the Boolean algebras $\{a, b\}$, with $a \leq b$, and the 6-tuple $\langle J(X), \cup, \cap, ^c, X, \emptyset\rangle$ where X is the real interval $[0, 1)$, $J(X)$ is the set of the finite unions of intervals of the form $[x, y)$, with $0 \leq x \leq y \leq 1$, and $A^c = X\setminus A$. We also consider $\langle J(X) \times \{a, b\}, \mathfrak{f}\rangle$, where $\mathfrak{f}(A, y) = (A, y)^{\intercal} = A^{\intercal} \times y^{\intercal}$, for every $(A, y) \in J(X) \times \{a, b\}$. By Proposition 4.3, it is an F_1-structure. The pairs $(\emptyset, a)$ and (X, b) are the zero and greatest elements, respectively.

Note that $(\emptyset, b)$ is the only atom in this Boolean algebra. This implies that $\mathcal{A} = \{(X, a)\}$. Hence, $\mathcal{A}\cap\uparrow\left(\left[0, \frac{1}{2}\right), b\right) = \emptyset$ because $\left(\left[0, \frac{1}{2}\right), b\right) \not\leq (X, a)$. Thus, by Lemma 4.6 (a) and Definition 4.5, $\langle J(X) \times \{a, b\}, \mathfrak{f}\rangle$ is not an atomic F_1-structure. Moreover, it is valid:

$$\bigcap_{(A,y)\in\mathcal{A}\cap\uparrow\left(\left[0,\frac{1}{2}\right),a\right)} \mathfrak{f}(A, y) = \mathfrak{f}(X, a) = J(X) \times \{b\}$$

However, $\mathfrak{f}\left(\left[0, \frac{1}{2}\right), a\right) = \left\{M \in J(X) : \left[0, \frac{1}{2}\right)^c \subseteq M\right\} \times \{b\}$. □

5. Concluding remarks

The main advantage of the new formulation of F_1-structures in this paper is that it is simpler and shorter than the original definition. So we can analyze examples and results associated with these models in a more practical way, as shown here. This not only encourages us to make a deeper study of the F_1-structures but also to find similar characterizations for the F-structures for the rest of the hierarchy $\{C_n\}_{n\in\mathbb{N}}$ of da Costa's logics.

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Instituto de Ciencias Básicas
Universidad Nacional de San Juan
(5400) San Juan, Argentina
e-mail: vquiroga@ffha.unsj.edu.ar