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BISIMULATION REDUCTS AND SUBMODELS OF INTUITIONISTIC FIRST-ORDER KRIPKE MODELS

Abstract

We consider elementary submodels of a given intuitionistic Kripke model $\mathcal{K}$ meant as models that share the same theory with $\mathcal{K}$ and result in restricting the frame of $\mathcal{K}$ and/or replacing some of its worlds with their elementary substructures. We introduce the notion of bisimulation reduct of the Kripke model wich allows us to construct elementary submodels of given Kripke models in the sense of the definition above.

As it was observed by A. Visser in [6], the notion of submodel can be defined for intuitionistic first-order Kripke models in several different ways. We can either consider models on the same frame, where the worlds of submodels are substructures of the worlds of the original model, or we can define a submodel to be the result of restricting the frame of the given model, or we can combine both of these operations. All of these possibilities were considered in the literature, see [1], [6] and [2] respectively, however it seems that we should accept the third notion as the correct one. The reason for that is, that not only such defined notion of submodel coincides with the classical notion of substructure in the case of the simplest Kripke model, but also because the well-known classical Tarski-Łoś preservation theorem concerning substructures becomes a particular case of the result proven in [2]; i.e. the class of the formulas that are preserved under Kripke submodels is the class of an intuitionistic variant of universal formulas.

In a natural way we can then ask about *elementary* submodels. However, one can easily notice that the operation of restricting the frame changes, in general, the theory of the given Kripke model since, for example, any one-node submodel of a given Kripke model always validates the Law of Excluded Middle. Furthermore, even when we consider submodels over the same frame as the given Kripke model $\mathcal{K}$, whose worlds are elementary substructures (in the classical sense) of the corresponding worlds of $\mathcal{K}$, we do not necessarily get an elementary Kripke submodel of $\mathcal{K}$. As a simple example consider two classical structures M and N that are models of true arithmetic, and a homomorphism $f\colon \mathrm{M} \to \mathrm{N}$ which is not elementary. Now, let $\mathcal{K}$ be the Kripke model whose worlds are M and N and whose only morphism is f. Since the standard model of arithmetic $\mathbb{N}$ is a prime model of true arithmetic, it is an elementary substructure of the models M and N. So, the model $\mathcal{N}$, consisting of two copies of the standard model and identity as a morphism, is a submodel of $\mathcal{K}$. But, of course, $\mathcal{N}$ validates the classical theory of true arithmetic, while $\mathcal{K}$ does not. So, although its worlds are elementary substructures of the corresponding worlds of $\mathcal{K}$, the model $\mathcal{N}$ cannot be elementary submodel of $\mathcal{K}$. These examples show clearly that if we want to construct an elementary submodel of a given Kripke model, we must take into account also the properties of the morphisms of the model in question.

From the characterization of partially elementary models given in [4], it follows that any Γ-elementary extension Kripke model $\mathcal{K}$, viewed as the submodel of $\mathcal{K}$, is a Γ-elementary submodel of $\mathcal{K}$. Of course, the models considered force strong intuitionistically not valid principles as Law of Excluded Middle. In this paper we will show how to construct elementary submodels that need not force any instances of the Law of Excluded Middle and any other non-intuitionistic principles.

Throughout this paper we fix a first-order language L of the usual logical operators $\bot$, $\wedge$, $\vee$, $\to$ and quantifiers $\exists$ and $\forall$. Following [6], we define a *Kripke model* as a functor $\mathcal{K} : \mathbb{K} \to \mathbb{S}$ from a partial order $\mathbb{K}$, considered as an order category, to the category $\mathbb{S}$ of classical first-order structures with weak homomorphisms as its arrows. We write $\alpha \leq^f \alpha'$ if f is an arrow from α to α'. Traditionally, the category $\mathbb{K}$ is called the *frame*, and its objects are called the *nodes* of $\mathcal{K}$ and classical structures assigned to the nodes are called the *worlds* of the model $\mathcal{K}$. To keep the notation simple, we will confuse an arrow f with its corresponding homomorphism $\mathcal{K}(f)$. Note that by $\mathcal{K}(k)$ we denote the *classical first-order structure* that is assigned to a node k. A *pointed model* $(\mathcal{K}, k)$ is a model $\mathcal{K}$ with a distinguished node k.

The forcing relation $\Vdash$ is defined in the usual way. For every model $\mathcal{K}$, its node k and a sequence of elements $\bar{a}$ of the world $\mathcal{K}(k)$, we put inductively,

$(\mathcal{K}, \alpha) \nVdash \bot$,

$(\mathcal{K}, \alpha) \Vdash P[\bar{a}]$ iff $\mathcal{K}(\alpha) \models P[\bar{a}]$, where $P(\bar{x})$ is atomic,

$(\mathcal{K}, \alpha) \Vdash (A \wedge B)\ [\bar{a}]$ iff $(\mathcal{K}, \alpha) \Vdash A[\bar{a}]$ and $(\mathcal{K}, \alpha) \Vdash B[\bar{a}]$,

$(\mathcal{K}, \alpha) \Vdash (A \vee B)\ [\bar{a}]$ iff $(\mathcal{K}, \alpha) \Vdash A[\bar{a}]$ or $(\mathcal{K}, \alpha) \Vdash B[\bar{a}]$,

$(\mathcal{K}, \alpha) \Vdash (A \to B)\ [\bar{a}]$ iff for all α' such that $\alpha \leq^f \alpha'$, whenever $(\mathcal{K}, \alpha') \Vdash A[f\bar{a}]$ then $(\mathcal{K}, \alpha') \Vdash B[f\bar{a}]$,

$(\mathcal{K}, \alpha) \Vdash (\exists x A(x))\ [\bar{a}]$ iff $(\mathcal{K}, \alpha) \Vdash A(x)[\bar{a}b]$, for some $b \in \mathcal{K}(\alpha)$,

$(\mathcal{K}, \alpha) \Vdash (\forall x A(x))\ [\bar{a}]$ iff $(\mathcal{K}, \alpha') \Vdash A(x)[f(\bar{a})b]$, for all α' such that $\alpha \leq^f \alpha'$ and $b \in \mathcal{K}(\alpha')$.

Let $\mathcal{K} : \mathbb{K} \to \mathbb{S}$ and $\mathcal{M} : \mathbb{M} \to \mathbb{S}$ be Kripke models. We say that $\mathcal{K}$ is a *submodel* of $\mathcal{M}$ if $\mathbb{K}$ is a subcategory of $\mathbb{M}$ and for every α in $\mathbb{K}$, the world $\mathcal{K}(\alpha)$ is a substructure of the world $\mathcal{M}(\alpha)$. For the sake of simplicity, we state the definition of elementary submodel for rooted models only. Let $\mathcal{K} : \mathbb{K} \to \mathbb{S}$ and $\mathcal{M} : \mathbb{M} \to \mathbb{S}$ be Kripke models with the roots α_0 and β_0, respectively. We say that $\mathcal{K}$ is an *elementary submodel of* $\mathcal{M}$ if

(1) $\mathcal{K}$ is a submodel of $\mathcal{M}$, and

(2) $(\mathcal{K}, \alpha_0) \Vdash A[\bar{a}]$ iff $(\mathcal{M}, \beta_0) \Vdash A[\bar{a}]$, for every finite sequence $\bar{a}$ from $\mathcal{K}(\alpha)$, and for every formula $A(\bar{x})$.

Finally, let us recall the notion of first-order bisimulation.

Let $\mathcal{K} : \mathbb{K} \to \mathbb{S}$ and $\mathcal{M} : \mathbb{M} \to \mathbb{S}$ be Kripke models. The symbols α, α' and β, β' range over the nodes of $\mathcal{K}$ and $\mathcal{M}$, respectively; π ranges over the collection of finite maps between worlds of $\mathcal{K}$ and worlds of $\mathcal{M}$. Now assume that $\pi : \mathcal{K}(\alpha) \to \mathcal{M}(\beta)$ and $\pi = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ for some $a_1, \ldots, a_n \in \mathcal{K}(\alpha)$ and $b_1, \ldots, b_n \in \mathcal{M}(\beta)$. In such a case we will also denote π by $(\bar{a}\,; \bar{b})$, where $\bar{a} = a_1, \ldots, a_n$ and $\bar{b} = b_1, \ldots, b_n$. If $\alpha \leq^f \alpha'$ and $\beta \leq^g \beta'$, then by $\pi^{f,g}$ we mean the map $\{(fa_1, gb_1), \ldots, (fa_n, gb_n)\}$, i.e. $\pi^{f,g} = (f\bar{a}\,; g\bar{b})$.

A *(first-order) bisimulation* between Kripke models $\mathcal{K}$ and $\mathcal{M}$ is a ternary relation which satisfies the conditions below. We will write $\pi : \alpha \sim \beta$ when π, α, β are in that relation.

(1) $\pi : \alpha \sim \beta$ implies that π is a partial isomorphism between $\mathcal{K}(\alpha)$ and $\mathcal{M}(\beta)$.

(2) $\pi : \alpha \sim \beta$ implies that π is a map between $\mathcal{K}(\alpha)$ and $\mathcal{M}(\beta)$ and
- ($\rightarrow$-zig) for every $\alpha \leq^f \alpha'$ there is $\beta \leq^g \beta'$ such that $\pi^{f,g} : \alpha' \sim \beta'$.
- ($\rightarrow$-zag) for every $\beta \leq^g \beta'$ there is $\alpha \leq^f \alpha'$ such that $\pi^{f,g} : \alpha' \sim \beta'$.

(3) $\pi : \alpha \sim \beta$ implies that π is a map between $\mathcal{K}(\alpha)$ and $\mathcal{M}(\beta)$ and
- ($\forall$-zig) for every $\alpha \leq^f \alpha'$ and $a' \in \mathcal{K}(\alpha')$ there is $\beta \leq^g \beta'$ and $b' \in \mathcal{M}(\beta')$ such that and $\pi^{f,g} \cup (a', b') : \alpha' \sim \beta'$;
- ($\forall$-zag) for every $\beta \leq^g \beta'$ and $b' \in \mathcal{M}(\beta')$ there is $\alpha \leq^f \alpha'$ and $a' \in \mathcal{K}(\alpha')$ such that $\pi^{f,g} \cup (a', b') : \alpha' \sim \beta'$.

(4) $\pi : \alpha \sim \beta$ implies that π is a map between $\mathcal{K}(\alpha)$ and $\mathcal{M}(\beta)$ and
- ($\exists$-zig) for every $a \in \mathcal{K}(\alpha)$ there is $b \in \mathcal{M}(\beta)$ such that $\pi \cup (a, b) : \alpha \sim \beta$;
- ($\exists$-zag) for every $b \in \mathcal{M}(\beta)$ there is $a \in \mathcal{K}(\alpha)$ such that $\pi \cup (a, b) : \alpha \sim \beta$.

Note that one can generalize the condition (3) to adding finite sequences, then (2) is the empty sequence case. However, the condition (4) cannot be obtained from (3). The reason is that, considering ($\exists$-zig), when we choose f to be id_α we *must* choose g to be id_β. But according to (3), g can be chosen arbitrarily.

Let α and β be nodes of the given models $\mathcal{K}$ and $\mathcal{M}$. We say that they *bisimulate*, in symbols $\alpha \approx \beta$, if there is a bisimulation $\sim$, and a map π, such that $\pi : \alpha \sim \beta$. Two pointed models $\mathcal{K} = (\mathcal{K}_0, \alpha_0)$ and $\mathcal{M} = (\mathcal{M}_0, \beta_0)$ are said to bisimulate, in symbols $\mathcal{K} \approx \mathcal{M}$, if the nodes α_0 and β_0 do. Finally, we say that the models $\mathcal{K}$ and $\mathcal{M}$ bisimulate if for every node α of $\mathcal{K}$ there is a node β of $\mathcal{M}$ such that the pointed models $(\mathcal{K}, \alpha)$ and $(\mathcal{M}, \beta)$ bisimulate, and vice versa.

Let us note that every bisimulation is contained in a maximal one. It is also known that maximal bisimulations are equivalence relations on the Kripke models on which they are defined. So, to distinguish them, maximal bisimulations will be denoted by the symbol $\approx$. In particular, we can prove the following fact.

LEMMA 1. *Let α, β and γ be nodes of a Kripke model $\mathcal{K}$ and let $\bar{a}$, $\bar{b}$, $\bar{c}$ be sequences of elements of $\mathcal{K}(\alpha)$, $\mathcal{K}(\beta)$ and $\mathcal{K}(\gamma)$ respectively of the same length. Finally, let $\approx$ be a maximal bisimulation on $\mathcal{K}$. Then if $(\bar{a}\,;\bar{b}) : \alpha \approx \beta$ and $(\bar{b}\,;\bar{c}) : \beta \approx \gamma$ then $(\bar{a}\,;\bar{c}) : \alpha \approx \gamma$*

Below we state the main result about first-order bisimulation. Its proof can be found in [5].

THEOREM 2. *Let* $(\mathcal{K}, \alpha)$ *and* $(\mathcal{M}, \beta)$ *be rooted Kripke models with the roots* α *and* β *respectively, and such that*

$$(\bar{a}\,; \bar{b}) : (\mathcal{K}, \alpha) \sim (\mathcal{M}, \beta),$$

for some finite mapping $(\bar{a}\,; \bar{b})$ *between the worlds* $\mathcal{K}(\alpha)$ *and* $\mathcal{M}(\beta)$ *and a bisimulation* $\sim$*. Then for every formula* $A(\bar{x})$ *we have*

$$(\mathcal{K}, \alpha) \Vdash A[\bar{a}] \iff (\mathcal{M}, \beta) \Vdash A[\bar{b}].$$

In the sequel we will introduce the notion of bisimulation reduct of a given first order Kripke model. Our notion will share similar properties with that of collapse of a Kripke model for propositional logic. However, a collapse of a model $\mathcal{K}$ is the quotient model of $\mathcal{K}$ with respect to the bisimulation relation on the set of the nodes of $\mathcal{K}$. On the other hand, our bisimulation reducts will not be quotient models, but will result in a careful selection of worlds from equivalence classes to form a submodel of the model in question without changing its theory. A similar approach can be found in the selection method for modal logics. This kind of idea, although without explicit use of bisimulation, was also applied in [3] in defining selective submodels of the canonical models for intuitionistic predicate logic. The notion of bisimulation reduct refers only to structural properties of the Kripke models and their worlds, and does not involve logical notions which are used when we consider the selection method and selective submodels.

To be able to carry out our construction, we confine to Kripke models over trees. In this paper by a *tree* we mean a partial order $\langle \mathbb{K} \leq \rangle$ such that $\mathbb{K}$ is non-empty, contains the least element, and for every $\alpha \in \mathbb{K}$ the set of all its predecessors is well ordered.

Let $\mathcal{K}$ be a Kripke model over a tree $\langle \mathbb{K}, \leq \rangle$ with the root λ. Consider a maximal bisimulation $\approx$ on $\mathcal{K}$. Let $[K]_\approx = \{[\alpha]_\approx : \alpha \in \mathbb{K}\}$ and put, as usual,

$$[\alpha]_\approx \preceq [\beta]_\approx \quad \text{iff} \quad \alpha' \leq \beta', \text{ for some } \alpha' \in [\alpha]_\approx \text{ and } \beta' \in [\beta]_\approx.$$

In the sequel, when the bisimulation is fixed, we will skip the subscript $\approx$. Of course, the quotient frame $\langle [\mathbb{K}], \preceq \rangle$ of $\langle \mathbb{K}, \leq \rangle$ need not be a tree.

Now we define a function $\rho : [\mathbb{K}] \to \mathbb{K}$ in the following way

(i) $\rho[\lambda] := \lambda$,

(ii) Assume that $[\alpha] \succeq [\lambda]$ and that ρ is defined for $[\alpha]$. For every immediate successor $[\beta]$ of $[\alpha]$ we choose an element $\beta^* \in [\beta]$ such that there is a morphism f with $\rho[\alpha] \leq^f \beta^*$ and put $\rho[\beta] := \beta^*$.

Observe that we can always find a node β^* satisfying (ii). Indeed, since $[\beta]$ is an immediate successor of $[\alpha]$, we can find $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$ with $\alpha' \leq^{f'} \beta'$. We have $\rho[\alpha] \approx \alpha'$, hence by the zig-zag conditions of bisimulation, there is an element β^* such that $\rho[\alpha] \leq^f \beta^*$ and $\beta^* \approx \beta'$.

Let $\mathbb{K}^\rho$ be the set of all chosen elements, i.e., we put

$$\mathbb{K}^\rho := \{\rho[\alpha]_\approx : \alpha \in \mathbb{K}\}.$$

Note that $\langle \mathbb{K}^\rho, \leq\restriction \mathbb{K}^\rho\rangle$ is a subtree of the tree $\langle \mathbb{K} \leq\rangle$.

Definition 3. We define a *bisimulation reduct* of the model $\mathcal{K}$ with respect to the maximal bisimulation $\approx$ and the choice function ρ as Kripke submodel $\mathcal{K}^\rho$ of $\mathcal{K}$ generated by the frame $\langle \mathbb{K}^\rho, \leq\restriction \mathbb{K}^\rho\rangle$.

We will prove the following fact.

Theorem 4. *Let $\mathcal{K}$ be a Kripke model over a tree. Then every bisimulation reduct of a first-order Kripke model is an elementary submodel of that model.*

Proof: Let $\mathcal{K}$ be a Kripke model over a tree $\mathbb{K}$ with the root λ. Let us consider a bisimulation reduct $\mathcal{K}^\rho$ of the model $\mathcal{K}$ with respect to a maximal bisimulation $\approx$ and a choice function ρ.

Let $\delta \in \mathbb{K}$ and $\gamma \in \mathbb{K}^\rho$ and let $(\bar{a}\,;\bar{b})$ be a finite mapping from $\mathcal{K}(\delta)$ to $\mathcal{K}^\rho(\gamma)$. We define a relation $\simeq$ in the following way:

$$(\bar{a}\,;\bar{b}) : \gamma \simeq \delta \;\text{ iff }\; \delta = \rho[\gamma] \text{ and } (\bar{a}\,;\bar{b}) : \gamma \approx \delta. \tag{1}$$

We will show that $\simeq$ is a bisimulation between $\mathcal{K}$ and $\mathcal{K}^\rho$. We verify the condition ($\forall$-zig) only, the other conditions can be verified in a similar way.

Assume that $(\bar{a}\,;\bar{b}) : \gamma \simeq \delta$. Then $\delta = \rho[\gamma]$ and

$$(\bar{a}\,;\bar{b}) : \gamma \approx \rho[\gamma]. \tag{2}$$

Now, let γ' be a node such that $\gamma \leq^f \gamma'$ and let c be an element of $\mathcal{K}(\gamma')$. By (2), there is δ' such that $\delta \leq^g \delta'$ and an element b' of $\mathcal{K}(\delta')$ such that

$$(f(\bar{a})a'\,; g(\bar{b})b') : \gamma' \approx \delta'. \tag{3}$$

Consider the node $\delta^* = \rho[\gamma']$. Since both δ and δ^* belong to the $\approx$-reduct of $\mathcal{K}$, we have $\delta \leq^{g^*} \delta^*$. Moreover, it is clear that $\delta^* \in [\gamma']$. So γ', δ' and δ^* are in the same equivalence class. It follows that

$$(g'(\bar{b})\,; g^*(\bar{b})) : \delta' \approx \delta^* \tag{4}$$

Then, in particular, there is $b^* \in \mathcal{K}(\delta')$ such that

$$(g'(\bar{b})b'\,; g^*(\bar{b})b^*) : \delta' \approx \delta^*. \tag{5}$$

Hence, by (3) and (5) and Lemma 1, we get

$$(f(\bar{a})a'\,; g^*(\bar{b})b^*) : \gamma' \approx \delta^*. \tag{6}$$

It follows that $(f(\bar{a})a'\,; g^*(\bar{b})b^*) : \gamma' \simeq \delta^*$, what was required. So, $\simeq$ is a bisimulation between the models $\mathcal{K}$ and $\mathcal{K}^\rho$.

Obviously, we have

$$(\bar{c}\,; \bar{c}) : (\mathcal{K}, \lambda) \simeq (\mathcal{K}^\rho, \rho[\lambda])$$

for every finite sequence $\bar{c}$ of the elements of $\mathcal{K}(\lambda)$. Hence, by Theorem 2, for every formula A we have

$$(\mathcal{K}, \lambda) \Vdash A[\bar{a}] \text{ iff } (\mathcal{K}^\rho, \rho[\lambda]) \Vdash A[\bar{a}].$$

So, the model $\mathcal{K}^\rho$ is an elementary submodel of the model $\mathcal{K}$. □

To state our results, we have to introduce an auxiliary notion. For simplicity, we refer it to models over the same frame. Let $\mathcal{K} : \mathbb{K} \to \mathbb{S}$ and $\mathcal{M} : \mathbb{K} \to \mathbb{S}$ be Kripke models over the same frame. A pair $(\bar{a}\,; \bar{b})$ of sequences of elements of $\mathcal{K}(\alpha)$ and and of $\mathcal{M}(\alpha)$ is an elementary map if for every formula $A(\bar{x})$ we have $\mathcal{K}(\alpha) \models A\,[\bar{a}]$ if and only if $\mathcal{M}(\alpha) \models A\,[\bar{b}]$. We say that an elementary map π between the classical structures $\mathcal{K}(\alpha)$ and $\mathcal{M}(\alpha)$ is *upwards preserved* iff for every $\alpha \leq^f \alpha'$, the map $\pi^{f,g}$ is also an elementary map between $\mathcal{K}(\alpha')$ and $\mathcal{M}(\alpha')$.

Now we can state a theorem which is a variant of [5, Theorem 4.4].

THEOREM 5. *Let* $\mathcal{K} : \mathbb{K} \to \mathbb{S}$ *and* $\mathcal{M} : \mathbb{K} \to \mathbb{S}$ *be rooted Kripke models over the same frame. Additionally, assume that*

(a) for every α*, the structure* $\mathcal{K}(\alpha)$ *is an elementary substructure of* $\mathcal{M}(\alpha)$*,*

(b) for every α*, all elementary maps between* $\mathcal{K}(\alpha)$ *and* $\mathcal{M}(\alpha)$ *are upwards preserved,*

(c) all the worlds of $\mathcal{K}$ *and* $\mathcal{M}$ *are* ω*-saturated.*

Then the Kripke model $\mathcal{K}$ *is an elementary submodel of the Kripke model* $\mathcal{M}$*.*

PROOF SKETCH: The proof goes along the lines of the proof of Theorem 4.4 of [5]. First we show that for each k and a tuple $(\bar{a}, \bar{a})$ in $\mathcal{K}(\alpha)$, the mapping $(\bar{a}, \bar{a}) : \mathcal{M}(\alpha) \to \mathcal{K}(\alpha)$ is a n-local-isomorphism, for every $n < \omega$. Then we show that for each α there is a bisimulation between the submodel of Kripke model $\mathcal{K}$ generated by α and the submodel of $\mathcal{M}$ generated by α. $\square$

Let us comment on other assumptions of the theorem. The assumption (a) seems to be natural. The condition (b) describes an anticipated property of morphisms. As we already know, a condition of this form is necessary. Finally, the assumption (c) is crucial for proving the existence of required bisimulations between the model $\mathcal{K}$ and its submodel, and in consequence, that the submodel in question is elementary. We do not know whether this assumption can be weakened or waved.

Theorem 5 suggests the operation which for a Kripke model $\mathcal{M}$ whose all the worlds are ω-saturated assigns a submodel $\mathcal{K}$ such that $\mathcal{K}$ and $\mathcal{M}$ satisfy the conditions (a), (b), (c). Any such a model $\mathcal{K}$ will be denoted by $\mathcal{M}^*$.

Now, a construction of an elementary Kripke submodel of a given model can be described in the two steps we showed above. Step One can be described as *removing the worlds* from a Kripke model in question. Step Two can be viewed as *taking elemntary substructures* of a given model. Note that in both steps the resulting model is an elementary submodel of the model in question. Now we can describe our construction of an elementary submodel.

Let $\mathcal{M}$ be a Kripke model over a tree such that all the worlds of $\mathcal{M}$ are ω-saturated and let $\approx$ be a maximal bisimulation on $\mathcal{M}$. Then we find a bisimulation reduct $\mathcal{M}^\rho$ of the model $\mathcal{M}$. Now, we choose the submodel $(\mathcal{M}^\rho)^*$ of $\mathcal{M}^\rho$ as in Theorem 5 such that $(\mathcal{M}^\rho)^* \prec \mathcal{M}$:

$$\mathcal{M} \xrightarrow{\rho} \mathcal{M}^\rho \xrightarrow{*} (\mathcal{M}^\rho)^* \prec \mathcal{M}$$

Note that our results can be generalized by considering *bounded* bisimulations, i.e. those that preserve the validity of formulas in a given class Γ only, by a suitable relativization of the assumptions (a), (b) and (c), for Γ-elementary submodels. In consequence, we can also construct Γ-elementary submodels according to the construction above.

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