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A NOTE ON POST-COMPLETE EXTENSIONS OF A LOGIC OF VALUES OF A. IVIN

Abstract

In this note we prove that the logic introduced by A. Ivin has infinitely many Post-complete extensions.

1. Modal logic.

A subset of the set of formulae L of the language of classical propositional logic (CPL) extended by the operator $\Box$ ("it's necessary that") is said to be a *regular* modal logic if L contains all propositional tautologies, the axiom

$K.\quad \Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$

closed under Modus Ponens and the Regularity Rule[1]. If L is closed under Necessitation Rule or contains a formula of the form $\Box\phi$, then L is said to be *normal.* The smallest normal modal logic is denoted by **K**. **L** is said to be a *strictly regular* modal logic if **L** is regular but is not normal. The smallest strictly regular logic is denoted by **C2** (cf. Lemmon [3]).

2. Ivin's logic of values.

A. Ivin described some logics of values (cf. Ivin [2], p. 124). The simplest of these logics, denoted by **GHm** is based on two operators: G ("It is good

[1]The axiom K can be replaced by K': $(\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.

that") and H ("It is bad that"). In the original version, the iteration of modalities G and H is not allowed. We generalize this logic by omitting this restriction. The logic **GHm** has the following axioms

A0. All tautologies of CPL in the language extended by G and H,
A1. $G(\phi \wedge \psi) \leftrightarrow G(\phi) \wedge G(\psi)$,
A2. $H(\phi \wedge \psi) \leftrightarrow H(\phi) \wedge H(\psi)$,
A3. $G(\phi) \rightarrow \neg H(\phi)$.

and is closed under
Modus Ponens: $\dfrac{\phi \rightarrow \psi,\ \phi}{\psi}$,

Extensionality Rules: (RE_G) $\dfrac{\phi \leftrightarrow \psi}{G(\phi) \leftrightarrow G(\psi)}$, (RE_H) $\dfrac{\phi \leftrightarrow \psi}{H(\phi) \leftrightarrow H(\psi)}$.

It follows that **GHm** is closed under the regularity rules

$$(RR_G)\ \frac{\phi \rightarrow \psi}{G(\phi) \rightarrow G(\psi)},\ (RR_H)\ \frac{\phi \rightarrow \psi}{H(\phi) \rightarrow H(\psi)}.$$

Note that logic **GHm** is stricly regular with respect to both modalities.

3. Algebraic semantics.

A *modal algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, \Box, 1, 0\rangle$, where the reduct $\langle A, \wedge, \vee, \neg, 1, 0\rangle$ is a Boolean algebra and $\Box$ satisfies the identity:
$\Box(x \wedge y) = \Box x \wedge \Box y$.

Ivin's logic is complete with respect to the so-called Ivin's algebras.

Definition 1. An algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, G, H, 1, 0\rangle$ is said to be an *Ivin's algebra* if the reduct $\langle A, \wedge, \vee, \neg, 1, 0\rangle$ is a Boolean algebra and functions G and H satisfy the identities:

(i) $G(x \wedge y) = G(x) \wedge G(y)$
(ii) $H(x \wedge y) = H(x) \wedge H(y)$
(iii) $G(x) \leq \neg H(x)$

The inequality (iii) is equivalent to the identity $G(x) \wedge H(x) = 0$.

Corollary 1. *For any modal algebra there exists at least one Ivin's algebra.*

PROOF: Let **A** be a modal algebra. The function $\Box$ denoted by G. If $G(1) = 1$, then we define a new function H in this algebra by taking $H(x) = 0$ for each x.

Let the algebra **A** be stricly regular (i.e. $\Box(1) \neq 1$). Then we can take $G = \Box$ and define a second modal function H such that $G(x) \wedge H(x) = 0$.

COROLLARY 2. *Each of Ivin's algebra is a "join" of two modal algebras.*

By construction of Lindenbaum algebra, it is easy to prove completeness the theorem for the logic **GHm**.

THEOREM 3 (COMPLETENESS OF GHM.) $\vdash_{GH} \phi$ *iff* $h^v(\phi) = 1_{\mathbf{B}}$ *for any valuation* h^v *into any Ivin's algebra* **B**.

4. The variety of Ivin's algebras.

It follows from the definition of Ivin's algebras that the class of Ivin's algebras is a *variety*, i.e. it is closed under subalgebras, homomorphic images and products. By completeness theorem of Ivin's logic we conclude that axiomatic extensions of Ivin's logic are generated by subvarieties of the variety of all Ivin's algebras.

There are no problems here with subalgebras and products. We characterize now homomorphic images: congruences are generated here by filters.

Let F be a filter on Boolean algebra **A** and let $x, y \in F$. Let the relation $\sim_F \subseteq A \times A$ be defined by

$$x \sim_F y \text{ iff exist } z \in F \text{ so that } x \wedge z = y \wedge z.$$

It is well known that this relation is a congruence in this Boolean algebra **A** i.e. is a congruence with respect to the function $\neg, \wedge, \vee$.

Consider now a modal algebra **A**. A filter F on **A** is said to be *nearly-open* if it satisfies the implication

$$\text{if } x \in F \text{ then } \neg\Box(1) \vee \Box x \in F.$$

It is known that the lattice of congruences of a modal algebra is isomorphic to the lattice of nearly-open filters of this algebra (cf. Swirydowicz [4]).

This connection can be generalized to Ivin's algebras.

DEFINITION 2. Let **A** be a Ivin's algebra and let F be a filter on **A**. Then F is called a *nearly-open G-H filter* if F satisfies the following two conditions

1) if $x \in F$, then $\neg G(1) \vee G(x) \in F$
2) if $x \in F$, then $\neg H(1) \vee H(x) \in F$

Lemma 4. *Each congruence in Ivin's algebra* **A** *determines a nearly-open G-H filter on* **A**.

Proof: Let θ be a congruence relation on **A**. It is known that $F_\theta = \{x : \langle x, 1\rangle \in \theta\}$ is a filter on **A**. We prove that it is nearly-open.

Let $x \in F_\theta$, i.e. $\langle x, 1\rangle \in \theta$, thus $\langle Gx, G1\rangle \in \theta$ and $\langle \neg G1 \vee Gx, \neg G1 \vee G1\rangle \in \theta$, i.e. $\langle \neg G1 \vee Gx, 1\rangle \in \theta$. Thus $\neg G1 \vee Gx \in F_\theta$. The proof for the function H is analogous.

Lemma 5. *Each nearly-open G-H filter on* **A** *determines a congruence relation on* **A**.

Proof: (cf. Swirydowicz [4]) Let F be a nearly open G-H filter on **A**. Denote by $\theta(F)$ the relation on **A** defined by

$$x \sim_{\theta(F)} y \text{ iff exist } z \in F \text{ such that } x \wedge z = y \wedge z.$$

To prove that $\theta(F)$ is a congruence on **A**, it suffices to show that $\theta(F)$ is a G-H congruence.

Then let $x \sim_{\theta(F)} y$, so there exists $z \in F$ such that $x \wedge z = y \wedge z$. Hence $G(x) \wedge G(z) = G(y) \wedge G(z)$. Note that the functions G and H are monotonic, i.e. if $x \leq y$ then $Gx \leq Gy$; similarly for H. Thus $G(v) \wedge \neg G(1) = 0$ for all $v \in A$.

We have: $G(x) \wedge (\neg G(1) \vee G(z)) = (G(x) \wedge G(z)) \vee (G(x) \wedge \neg G(1)) = G(y) \wedge G(z) = (G(y) \wedge \neg G(1)) \vee (G(y) \wedge G(z)) = G(y) \wedge (\neg G(1) \vee G(z))$. Thus $G(x) \sim_F G(y)$, because $(\neg G(1) \vee G(z)) \in F$.

The proof for the function H is analogous.

Claim 6. *There exists five two-element Ivin's algebras.*

Proof: Let $\mathbf{2} = \langle \{0, 1\}, \wedge, \vee, \neg, 1, 0\rangle$ be the two-element Boolean algebra. By $\mathbf{2}^+, \mathbf{2}^=, \mathbf{2}^-$ denote two-element modal algebras with the function $\Box$ defined as follows: $\Box 1 = \Box 0 = 1$ in $\mathbf{2}^+$, $\Box 1 = 1, \Box 0 = 0$ in $\mathbf{2}^=$ and $\Box 1 = \Box 0 = 0$ in $\mathbf{2}^-$.

Similarly, denote i.g is Latin term for example by $\mathbf{2}^{-,-}$ the Ivin's algebra with the functions G and H defined by $G1 = G0 = 0$ and $H1 = H0 = 0$.

Using analogous notation we can describe all the remaining two-element Ivin's algebras: $\mathbf{2}^{-,=}, \mathbf{2}^{=,-}, \mathbf{2}^{-,+}, \mathbf{2}^{+,-}$.

5. Main result.

Now we construct an infinite sequence of Ivin's algebras which are zero-generated and have only two homomorphic images.

To simplify the construction, we define a "possibility" function:

$$g(x) \Leftrightarrow \neg G(\neg x)$$

It is easy to check that this function satisfy the equality:

$A1'$ $\quad g(x \vee y) = g(x) \vee g(y)$

Let us denote by $\mathbf{A}_n$ a Boolean algebra with n atoms $a_1, ..., a_n$ and let $1, 0$ be the unit and the zero of this algebra. We define function g on atoms of algebra $\mathbf{A}_n$ as follows:

$$g(x) = \begin{cases} a_1, & \text{for } x = 0 \\ a_1 \vee a_k \vee a_{k+1}, & \text{for } x = a_k, k \leq n-1 \\ a_1 \vee a_2 \vee a_n, & \text{for } x = a_n \end{cases}$$

and extend g to all elements of this algebra using the identity identity $A1'$: $g(x \vee y) = g(x) \vee g(y)$. Since $\mathbf{A}_n$ is finite, each of its elements is a join of a finite number of atoms, hence g will be defined for all x of $\mathbf{A}_n$.

Since g satisfies the identity $A1'$ and $g(0) \neq 0$, then g is a possibility function in a strictly regular modal algebra.

Let $\leq$ be a partial order on $\mathbf{A}_n$; let $<$ be a strict order defined by $x < y \Leftrightarrow x \leq y \wedge x \neq y$.

LEMMA 7. *For each x of $\mathbf{A}_n$, if $x \neq 1$ then $x < g(x)$.*

PROOF: Let $x = a_{i_1} \vee ... \vee a_{i_k}$ $(i_1 < ... < i_k)$. We have the following three cases:

a) $a_1 \neq a_{i_1}$. Then $x < g(x)$.

b) $a_1 = a_{i_1}, a_{i_k} \neq a_n$. Then $g(x) = g(a_{i_1} \vee ... \vee a_{i_k}) = g(a_{i_1}) \vee ... \vee g(a_{i_k}) = a_1 \vee (a_{i_1} \vee a_{i_1+1}) \vee ... \vee (a_{i_k} \vee a_{i_k+1})$. It is easy to observe that $x \leq g(x)$.

By the assumption $a_{i_k} \neq a_n$, then $g(a_{i_k}) = a_1 \vee a_{i_k} \vee a_{i_{k+1}}$, thus g adds to x at least the atom $a_{i_{k+1}}$. In consequence $x < g(x)$.

c) $a_1 = a_{i_1}, a_{i_k} = a_n$. By the assumption, $x \neq 1$. It means that the sequence of atoms $a_{i_1}, a_{i_2}, \ldots a_{i_k}$ from which element x is built does not contain all atoms. So there exists a ,,gap" in this sequence, i.e. there exists an atom a_{i_t} such that the a_{i_t+1} does not appear in x. It follows from the definition of g that a_{i_t+1} will be added to x by the function g. Thus $x < g(x)$ and it finishes the proof.

Now we can define in $\mathbf{A}_n$ a necessity function G by the identity $Gx = \neg g \neg x$. This function satisfies the equality $G(x \wedge y) = G(x) \wedge G(y)$, but $G1 \neq 1$, because $G1 = \neg g \neg 1 = \neg g0 = \neg a_1 = a_2 \vee \ldots \vee a_n$. Moreover, by Lemma 7, $G(x) < x$, provided that $x \neq 0$.

Now we define a function H as follows

$$H(x) = \begin{cases} a_1, & \text{for } x = 1 \\ 0, & \text{in other cases} \end{cases}$$

This function is a modal necessity function in a strictly regular modal algebra for any x $Gx \wedge Hx = 0$, so $Hx \leq \neg Gx$. Therefore the Boolean algebra $\mathbf{A}_n$ with the functions G and H is a Ivin's algebra. Since $\neg G1 = H1, \neg H1 = G1$, hence for any x of $\mathbf{A}_n$ either $x \leq G1$ or $H1 \leq x$.

Lemma 8. *There does not exists a proper subalgebra of any algebra $\mathbf{A}_n$ with the function g.*

Proof: Let the algebra $\mathbf{A}_n$ be fixed and 0 be the zero of $\mathbf{A}_n$. We have: $g(0) = a_1$, $g(a_1) = a_1 \vee a_2$, $g(a_1 \vee a_2) = a_1 \vee a_2 \vee a_3$,..., $g(a_1 \vee \ldots \vee a_{n-1}) = 1$, i.e. $g(0) = a_1$, $g^2(0) = g(g(0)) = a_1 \vee a_2$, $g^3(0) = g(g(g(0))) = a_1 \vee a_2 \vee a_3$ etc. It is easy to observe that $g^2(0) \setminus g(0) = a_2$, $g^3(0) \setminus g^2(0) = a_3$,..., $g^n(0) \setminus g^{n-1}(0) = a_n$. Hence the algebra $\mathbf{A}_n$ with the function g is zero-generated.

So, if $\mathbf{B}$ is a subalgebra of $\mathbf{A}_n$, then $\mathbf{B}$ contains zero, and in therefore B contains all the elements of $\mathbf{A}_n$.

Lemma 9. *There does not exist a nontrivial homomorphic image of any algebra $\mathbf{A}_n$ with functions G and H.*

Proof: Each homomorphic image of Ivin's algebra is determined by a nearly-open G-H filter. Since the algebra for any x is finite, each filter is principal. Since for any x: either $H1 \leq x$, or $x \leq G1$, then it suffices to consider two cases.

a) Let $H1 \leq x_0$, $x_0 \neq 1$ and let $[x_0)$ be a filter generated by x_0. If a congruence is determined by $[x_0)$, then $\neg H1 \vee Hx_0 \in [x_0)$, i.e. $x_0 \leq \neg H1 \vee Hx_0$. Thus $x_0 \leq \neg a_1 \vee 0$, and which means that $x_0 \leq a_2 \vee ... \vee a_n$, but it is impossible, because $a_2 \vee ... \vee a_n \leq G1$.

b) Let $x_0 \leq G1$, $x_0 \neq 0$. Let $[x_0)$ determine a congruence. Then, as in the case a), $x_0 \leq \neg G1 \vee Gx_0$, i.e. $x_0 \leq a_1 \vee Gx_0$. By the assumption neither x_0 nor Gx_0 contains a_1. Therefore $x_0 \wedge \neg a_1 \leq Gx_0 \wedge \neg a_1$, so $x_0 \leq Gx_0$, and this is impossible (Lemma 7).

By Lemma 5 each congruence in Ivin's algebra is determined by a nearly-open G-H filter. Since $\mathbf{A}_n$-algebras have no nontrivial nearly-open G-H filters, they have no non-trivial congruences. Thus each $\mathbf{A}_n$-algebra is simple, i.e. has only two (trivial) congruences.

Theorem. *There exists countably infinitely many Post-complete extensions of the Ivin's logic* **GHm**.

Proof: Consider the sequence of algebras $\mathbf{A}_n$ ($n = 2, 3, \ldots$) with functions G and H. In fact, by Lemma 8 no algebra $\mathbf{A}_n$ has a proper subalgebra. Moreover, by Lemma 9 each algebra $\mathbf{A}_n$ is simple. It finishes the proof.

In [2] A. Ivin presented some axiomatic extensions of the basic logic of values we have analyzed above. On the one hand he added new axioms characterizing G and H (e.g. $G(\phi) \rightarrow \neg G(\neg\phi)$, $H(\phi) \rightarrow \neg H(\neg\phi)$). On the other hand he introduced a modal necessity function of indifference, I, ("It is indifferent that") and connect this function with G and H by adding the axiom: everything is good, bad or indifferent. However, from the point of view of algebra these extensions does not deliver any seriously interesting mathematical problems.

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