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MINIMAL SEQUENT CALCULI FOR MONOTONIC CHAIN FINITELY-VALUED LOGICS*

Abstract

A *monotonic* chain finitely-valued logic is an abstract algebra whose carrier is a finite ordinal $n \geqslant 2$ and whose operations are either monotonic or anti-monotonic with respect to the natural ordering on n for each argument. This notion covers Dunn's finitely-valued normal extensions of RM [1] as well as finitely-valued Gödel's [2] and Łukasiewicz's [3] logics. The paper provides an effective construction of a $2(n-1)$-place sequent calculus (in the sense of [7, 6]) with cut-elimination property and a strong completeness with respect to the logic involved which is most compact among similar calculi in the sense of a complexity of systems of premises of introduction rules. The computational complexity of the procedure of generating the mentioned calculi is polynomial on n. We exemplify our general approach by presenting analytical expressions of calculi for Dunn's finitely-valued normal extensions of RM and finitely-valued Gödel's logics.

Keywords: Sequent calculus, finitely-valued logic.

1. General background

We treat natural numbers as ordinals, that is, sets of lesser natural numbers (including 0). The power set of a set S is denoted by $\wp(S)$. Given a finite set S and a partial ordering $\leqq$ on S, the set of all elements of S minimal

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with respect to $\leqq$ is denoted by $\mathrm{Min}_{\leqq}(S)$. Further, S is referred to as an *anti-chain (with respect to* $\leqq$*)* whenever $\mathrm{Min}_{\leqq}(S) = S$. (Notice that $\mathrm{Min}_{\leqq}(S)$ is an anti-chain with respect to $\leqq$.) Given any $T \subseteq S$, put $T{\Uparrow}_{\leqq} \triangleq \{a \in S \mid \exists b \in T : b \leqq a\}$.

Abstract algebras are denoted by capital Gothic letters (possibly with indices), their carriers being denoted by corresponding Italic letters (with the same indices, if any). Likewise, first-order structures are denoted by capital Calligraphic letters (possibly with indices), their underlying algebras being denoted by corresponding Gothic letters (with the same indices, if any). Given a first-order structure $\mathcal{A}$ of signature Σ and its subsignature $\Sigma' \subseteq \Sigma$, the Σ'-reduct of $\mathcal{A}$ is denoted by $\mathcal{A}{\upharpoonright}\Sigma'$.

Concerning signed sequent calculi[1], we entirely follow the formalism developed in [4] in addition involving a useful first-order interpretation of signed sequents and rules for contracting our exposition of necessary semantic conceptions and facilitating readers' comprehension of the material.

Any finite propositional language L is traditionally treated as a functional signature, propositional L-formulas being considered as L-terms with variables from the countable set $\mathrm{Var} \triangleq \{p_i\}_{i\in\omega}$ of propositional variables. (The set of all propositional L-formulas is denoted by Fm_L.) Further, signs are treated as unary relation symbols. Then, given a finite set of signs S, we have a first-order signature $L \cup S$ while any *S-signed L-formula* $\Phi = s{:}\phi$, where $s \in S$ and $\phi \in \mathrm{Fm}_L$, is identified with the atomic first-order formula $s(\phi)$. (Given any $T \subseteq S$ and any $F \subseteq \mathrm{Fm}_L$, put $T{:}F \triangleq \{s{:}\phi \mid s \in T, \phi \in F\}$.) In this way, any *$S$-signed L-sequent* Φ being a finite set of S-signed L-formulas is identified with the multiple disjunction $\bigvee \Phi$. (The set of all variables actually occurring in Φ is denoted by $\mathrm{Var}(\Phi)$.) Likewise, any *S-sequent L-rule* $\frac{\Gamma}{\Phi}$, where $\Gamma \cup \{\Phi\}$ is a finite set of S-signed L-sequents, is identified with the formula $\bigwedge \Gamma \to \Phi$. Finally, an *$S$-signed L-matrix* is any first-order $(L \cup S)$-structure. Then, the concepts of the validity of signed formulas, sequents and sequent rules in a signed matrix (under an interpretation in it) become perfectly clear.

As for the conception of *derivation in* an *S-sequent L-calculus* $\mathcal{C}$ (*i.e.*, a set of S-sequent L-rules), it is defined in an almost standard manner except that not merely rules of $\mathcal{C}$ but also arbitrary multiplications of ar-

[1]As usual, we treat places as signs.

bitrary substitutional instances of rules of $\mathcal{C}$ are used for construction of derivations.[2]

Subsets of $S{:}\{p_0\}$ are referred to as *unitary* (signed sequents or axioms). The only universal *structural* S-sequent rules are *Weakenings* $\frac{\{\varnothing\}}{\{s{:}p_0\}}$, where $s \in S$. Given any $T \subseteq S$, we have both the (unitary) T-*axiom* $T{:}\{p_0\}$ and the T-*Cut* $\frac{\{\{s{:}p_0\} \mid s \in T\}}{\{\varnothing\}}$. Given any $s \in S$ and any $\gamma \in L$ of arity $k \in \omega$, an *introduction rule (of type $s : \gamma$)* is any S-sequent L-rule of the form $\frac{\Gamma}{\{s{:}\gamma(p_i)_{i \in k}\}}$ where $\Gamma \subseteq \wp(S{:}\{p_i\}_{i \in k})$.

Let $\mathcal{M}$ be an S-signed L-matrix. An introduction rule $\frac{\Gamma}{\Phi}$ is said to be *for* $\mathcal{M}$ provided

$$\mathcal{M} \models (\bigwedge \Gamma \leftrightarrow \Phi) \tag{1}$$

Further, $\mathcal{M}$ is said to be *singular* whenever, for each $a \in M$, there is some $T \subseteq S$ such that $\bigcap_{s \in T} s^{\mathcal{M}} = \{a\}$. A mapping $\nu : S \to \wp(\wp(S))$ is referred to as an *S-signed complement for* $\mathcal{M}$ if, for each $s \in S$, $M \setminus s^{\mathcal{M}} = \bigcap\{\bigcup\{q^{\mathcal{M}} \mid q \in Q\} \mid Q \in \nu(s)\}$.[3]

2. Main results

Fix any $n \in \omega \setminus 2$, any propositional language L and any L-algebra $\mathfrak{A}$ with carrier n. Consider any $\lambda \in L$ of arity $k \in \omega$. The operation $\lambda^{\mathfrak{A}}$ is said to be *(K-)monotonic*, where $K \subseteq k$, if, for each $j \in K$ [$\in k \setminus K$], every $\bar{c} \in n^j$, every $\bar{d} \in d^{k-1-j}$ and all $a, b \in n$ such that $a \leqslant b$, it holds that $\lambda^{\mathfrak{A}}(\bar{c}, a, \bar{d}) \leqslant [\geqslant] \lambda^{\mathfrak{A}}(\bar{c}, b, \bar{d})$.[4] Then, $\mathfrak{A}$ is referred to as *monotonic* provided all its operations are monotonic.

From now on, $\mathfrak{A}$ is supposed to be monotonic.

For all $i \in n \setminus 1$, put $F_i \triangleq \{l \in n \mid l \geqslant i\}$ (these are all non-empty proper filters of the natural ordering on n). Likewise, for all $j \in n - 1$,

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[2]Such formalism adopted in [4] provides most compact representation of multiplicative calculi and makes proposed calculi for finitely-valued logics really finite.

[3]This notion is due to [5].

[4]As for Dunn's, Łukasiewicz's and Gödel's finitely-valued logics, their negations are $\varnothing$-monotonic, conjunctions and disjunctions are 2-monotonic while implications are $\{1\}$-monotonic.

set $I_j \triangleq \{m \in n \mid m \leqslant j\}$ (these are all non-empty proper ideals of the natural ordering on n). Here, we deal with the following set $S_n \triangleq \{F_i \mid i \in n \setminus 1\} \cup \{I_j \mid j \in n-1\}$ of $2(n-1)$ signs. In this way, we have the *associated* S_n-signed L-matrix $\mathcal{M}^{\mathfrak{A}}$, where $\mathcal{M}^{\mathfrak{A}} \restriction L \triangleq \mathfrak{A}$ and $s^{\mathcal{M}^{\mathfrak{A}}} \triangleq s$ for all $s \in S_n$.

LEMMA 2.1. *$\mathcal{M}^{\mathfrak{A}}$ is singular.*

PROOF: For any $i \in (n-1) \setminus 1$, $F_i \cap I_i = \{i\}$. Finally, $I_0 = \{0\}$ and $F_{n-1} = \{n-1\}$. □

REMARK 2.2. It is clear that the mapping $\nu_n : S_n \to \wp(\wp(S_n)), s \mapsto \{\{n \setminus s\}\}$ is an S_n-signed complement for $\mathcal{M}^{\mathfrak{A}}$.

Next, we show how one can construct signed sequent calculi for associated signed matrices. We start from analyzing unitary axioms.

LEMMA 2.3. *Let $T \subseteq S$. The T-axiom is valid in $\mathcal{M}^{\mathfrak{A}}$ iff there are some $i \in n \setminus 1$ and some $j \in n-1$ such that $i-1 \leqslant j$ and $\{F_i, I_j\} \subseteq T$.*

PROOF: The "if"-part is obvious. Concerning the converse, suppose that, for all $i \in n \setminus 1$ and all $j \in n-1$ such that $\{F_i, I_j\} \subseteq T$ it holds that $i-1 > j$. In case $I \triangleq \{k \in n \setminus 1 \mid F_k \in T\} = \varnothing$, the T-axiom is not valid in $\mathcal{M}^{\mathfrak{A}}$ under the interpretation $p_0 \mapsto n-1$ in it for no proper ideal contains $n-1$. Finally, assume $I \neq \varnothing$. Set $i \triangleq \min I$. Since $F_i \in T$, for all $j \in n-1$ such that $I_j \in T$ it holds that $i-1 > j$. Then, $i-1 \notin s$ for each $s \in T$. Therefore, the T-axiom is not valid in $\mathcal{M}^{\mathfrak{A}}$ under the interpretation $p_0 \mapsto i-1$ in it. □

Put $\mathrm{Ax}_n \triangleq \{\{F_i{:}p_0, I_j{:}p_0\} \mid i \in n \setminus 1, j \in n-1, i-1 \leqslant j\}$. As an immediate consequence of Lemma 2.3, we have

COROLLARY 2.4. *Ax_n is the set of all minimal (by inclusion) unitary axioms valid in $\mathcal{M}^{\mathfrak{A}}$.*

By Lemma 2.1, Corollary 2.4 and [4], we have the following particular case of the Weak Completeness Theorem to be useful for further argumentation in its own right.

Corollary 2.5. *Any S_n-signed $\varnothing$-sequent is derivable in* Ax_n *iff it is valid in $\mathcal{M}^{\mathcal{A}}$.*

Choose any $\lambda \in L$ of arity $k \in \omega$ and any $\sigma \in S_n$. Then, $\lambda^{\mathfrak{A}}$ is K-monotonic for some $K \subseteq k$. Put $N \triangleq K$, if σ is a filter, and $N \triangleq k \setminus K$, otherwise. Now, we are going to propose a minimal introduction rule for $\mathcal{M}^{\mathfrak{A}}$ of type $\sigma{:}\lambda$.

Let $V \subseteq \mathrm{Var}$. An S_n-signed L-sequent Δ is said to be *V-monotonic* if, for each $v \in V$ ($\in \mathrm{Var} \setminus V$) and any $h, g : \mathrm{Var} \to n$ such that $h(v) \leqslant g(v)$ and $h(u) = g(u)$ for all $u \in \mathrm{Var} \setminus \{v\}$, it holds that $\mathcal{M}^{\mathfrak{A}} \models \Delta[h] \Rightarrow$ $(\Leftarrow)\mathcal{M}^{\mathfrak{A}} \models \Delta[g]$. An S_n-signed $\varnothing$-sequent Δ is said to be *of (sharing) type* V if, for each $v \in V$, $\Delta \cap (\{I_j \mid j \in n-1\}{:}\{v\}) = \varnothing$ while, for each $u \in \mathrm{Var} \setminus V$, $\Delta \cap (\{F_i \mid i \in n \setminus 1\}{:}\{u\}) = \varnothing$, in which case Δ is V-monotonic and, by Corollary 2.5, is not valid in $\mathcal{M}^{\mathfrak{A}}$. An S_n-signed $\varnothing$-sequent Δ is said to be *functional* if the relation $\Re_\Delta \triangleq \{\langle v, s\rangle \mid s{:}v \in \Delta\}$ is a function, in which case Δ is of some type $V' \subseteq \mathrm{Var}$. Given two functional S_n-signed $\varnothing$-sequents Γ and Δ, we write $\Gamma \preccurlyeq \Delta$ and say that Γ *subsumes* Δ if $\mathrm{Var}(\Gamma) \subseteq \mathrm{Var}(\Delta)$ and, for each $v \in \mathrm{Var}(\Gamma)$, $\Re_\Gamma(v) \subseteq \Re_\Delta(v)$, in which case $\mathfrak{M}^{\mathfrak{A}} \models (\Gamma \to \Delta)$. Clearly, $\preccurlyeq$ is a partial ordering on the set of all functional S_n-signed $\varnothing$-sequents.

Remark 2.6. Since both $\{F_i \mid i \in n \setminus 1\}$ and $\{I_j \mid j \in n-1\}$ are chains by inclusion, for any S_n-signed $\varnothing$-sequent Δ of any type $V \subseteq \mathrm{Var}$, we have a functional S_n-signed $\varnothing$-sequent $\overrightarrow{\Delta} \triangleq \{\bigcup\{s \in S_n \mid s{:}v \in \Delta\}{:}v \mid v \in \mathrm{Var}(\Delta)\}$ of type V such that $\mathcal{M}^{\mathfrak{A}} \models (\overrightarrow{\Delta} \leftrightarrow \Delta)$.

Given a set Ξ of S_n-signed $\varnothing$-sequents of any type, put $\overrightarrow{\Xi} \triangleq \{\overrightarrow{\Phi} \mid \Phi \in \Xi\}$.

Lemma 2.7. *Let $V \subseteq \mathrm{Var}$, Ξ a finite set of V-monotonic S_n-signed L-sequents and Δ an S_n-signed $\varnothing$-sequent such that $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Xi \to \Delta)$ while $\mathcal{M}^{\mathfrak{A}} \not\models \Delta$. Then, there is some functional $\Gamma \subseteq \Delta$ of type V such that $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Xi \to \Gamma)$.*

Proof: We start from proving a weaker auxiliary statement.

Claim 2.8. *Let $V \subseteq \mathrm{Var}$, $v \in \mathrm{Var}$, Ξ a finite set of V-monotonic S_n-signed L-sequents and Δ an S_n-signed $\varnothing$-sequent such that $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Xi \to \Delta)$*

while $\mathcal{M}^{\mathfrak{A}} \not\models \Delta$*. Then, there is some* $\Gamma \subseteq \Delta$ *such that* $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Xi \to \Gamma)$ *and* $\Gamma \cap (S_n{:}\{v\})$ *is of type* V*.*

Proof: Assume $v \in V$. (The opposite case is considered in a quite dual manner.) First, suppose $J \triangleq \{j \in n-1 \mid I_j{:}v \in \Delta\} = \varnothing$. Then, $\Delta \cap (S_n{:}\{v\})$ is of type V. In that case putting $\Gamma \triangleq \Delta$ completes the argument. Finally, suppose $J \neq \varnothing$. Set $\Gamma \triangleq \Delta \setminus \{I_j{:}v \mid j \in J\}$. Clearly, $\Gamma \cap (S_n{:}\{v\})$ is of type V. We prove $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Xi \to \Gamma)$ by contradiction. Assume there is some $h : \mathrm{Var} \to n$ such that $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Xi)[h]$ while $\mathcal{M}^{\mathfrak{A}} \not\models \Gamma[h]$. Define a $g : \mathrm{Var} \to n$ as follows. Put $g(v) \triangleq \max(h(v), 1 + \max J)$ and $g(u) \triangleq h(u)$ for all $u \in \mathrm{Var} \setminus \{v\}$. As $\mathcal{M}^{\mathfrak{A}} \not\models \Delta$, by Corollary 2.5, we conclude that $\mathcal{M}^{\mathfrak{A}} \not\models \Delta[g]$. However, $h(v) \leqslant g(v)$. Hence, $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Xi)[g]$ for $v \in V$ while each member of Ξ is V-monotonic. This contradicts the fact that $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Xi \to \Delta)$. Thus, $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Xi \to \Gamma)$. □

Applying Claim 2.8 consecutively for each $v \in \mathrm{Var}(\Delta)$ and, then, Remark 2.6, we prove the statement of the lemma. □

Corollary 2.9. *For any* $\Delta \subseteq S_n{:}\{p_l\}_{l \in k}$ *such that* $\mathcal{M}^{\mathfrak{A}} \not\models \Delta$ *while* $\mathcal{M}^{\mathfrak{A}} \models (\{\sigma{:}\lambda(p_l)_{l \in k}\} \to \Delta)$*, there is some functional* $\Gamma \subseteq \Delta$ *of type* $\{p_l\}_{l \in N}$ *such that* $\mathcal{M}^{\mathfrak{A}} \models (\{\sigma{:}\lambda(p_l)_{l \in k}\} \to \Gamma)$*.*

Proof: Note that $\{\sigma{:}\lambda(p_l)_{l \in k}\}$ is $\{p_l\}_{l \in N}$-monotonic. Then, Lemma 2.7 completes the argument. □

Let $C^{\mathfrak{A}}_{\sigma:\lambda}$ be the set of all functional $\Delta \subseteq S_n{:}\{p_l\}_{l \in k}$ of type $\{p_l\}_{l \in N}$ such that $\mathcal{M}^{\mathfrak{A}} \models (\{\sigma{:}\lambda(p_l)_{l \in k}\} \to \Delta)$. Put $P^{\mathfrak{A}}_{\sigma:\lambda} \triangleq \mathrm{Min}_{\preccurlyeq} C^{\mathfrak{A}}_{\sigma:\lambda}$ and $R^{\mathfrak{A}}_{\sigma:\lambda} \triangleq \dfrac{P^{\mathfrak{A}}_{\sigma:\lambda}}{\{\sigma{:}\lambda(p_l)_{l \in k}\}}$.

Note 2.10. According to Remark 3.10 of [5], by Lemma 2.1, $Q^{\mathfrak{A}}_{\sigma:\lambda} \triangleq \{\bigcup_{l \in k}(\{s \in S_n \mid a_l \notin s\}{:}\{p_l\}) \mid \bar{a} \in n^k, \lambda^{\mathfrak{A}}(\bar{a}) \notin \sigma\}$ is the set of premises of an introduction rule for $\mathcal{M}^{\mathfrak{A}}$ of type $\sigma{:}\lambda$. Clearly, no member of $Q^{\mathfrak{A}}_{\sigma:\lambda}$ is valid in $\mathcal{M}^{\mathfrak{A}}$.

Theorem 2.11. $R^{\mathfrak{A}}_{\sigma:\lambda}$ *is an introduction rule for* $\mathcal{M}^{\mathfrak{A}}$*.*

PROOF: By Note 2.10, (1) and Corollary 2.9, for every $\Phi \in Q^{\mathfrak{A}}_{\sigma:\lambda}$, there is some $\Psi \in C^{\mathfrak{A}}_{\sigma:\lambda}$ such that $\Psi \subseteq \Phi$. Then, there is some $\Upsilon \in P^{\mathfrak{A}}_{\sigma:\lambda}$ such that $\Upsilon \preccurlyeq \Psi$, in which case $\mathcal{M}^{\mathfrak{A}} \models (\Upsilon \to \Phi)$. Hence, $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge P^{\mathfrak{A}}_{\sigma:\lambda} \to \bigwedge Q^{\mathfrak{A}}_{\sigma:\lambda})$. Therefore, by (1), we have $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge P^{\mathfrak{A}}_{\sigma:\lambda} \to \{\sigma{:}\lambda(p_l)_{l\in k}\})$. As the converse implication is trivial, we get (1) for $R^{\mathfrak{A}}_{\sigma:\lambda}$. □

LEMMA 2.12. *Let $V \subseteq \mathrm{Var}$ and $\Delta \cup \{\Phi\}$ a finite set of functional S_n-signed $\varnothing$-sequents of type V such that $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Delta \to \Phi)$. Then, $\Phi \in \Delta{\Uparrow}_{\preccurlyeq}$.*

PROOF: By contradiction. Suppose $\Phi \notin \Delta{\Uparrow}_{\preccurlyeq}$. Then, for each $\Psi \in \Delta$, there is some $v_\Psi \in \mathrm{Var}(\Psi)$ such that either $v_\Psi \notin \mathrm{Var}(\Phi)$, in which case choose any $a_\Psi \in \Re_\Psi(v_\Psi)$, or $v_\Psi \in \mathrm{Var}(\Phi)$ but $\Re_\Psi(v_\Psi) \not\subseteq \Re_\Phi(v_\Psi)$, in which case choose any $a_\Psi \in \Re_\Psi(v_\Psi) \setminus \Re_\Phi(v_\Psi)$. For each $v \in \{v_\Psi \mid \Psi \in \Delta\} \cap V$, put $g_v \triangleq \max\{a_\Psi \mid \Psi \in \Delta, v_\Psi = v\}$. Further, for every $v \in \{v_\Psi \mid \Psi \in \Delta\} \setminus V$, set $g_v \triangleq \min\{a_\Psi \mid \Psi \in \Delta, v_\Psi = v\}$. Consider any $h : \mathrm{Var} \to n$ such that, for all $v \in \{v_\Psi \mid \Psi \in \Delta\}$, $h(v) = g_v$ and, for all $v \in \mathrm{Var}(\Phi) \setminus \{v_\Psi \mid \Psi \in \Delta\}$, $h(v) \notin \Re_\Phi(v)$. Then, $\mathcal{M}^{\mathfrak{A}} \models \bigwedge \Delta[h]$ but $\mathcal{M}^{\mathfrak{A}} \not\models \Phi[h]$. This contradiction completes the argument. □

Lemmas 2.7 and 2.12 immediately yield

COROLLARY 2.13. *Let $V \subseteq \mathrm{Var}$, Δ a finite set of functional S_n-signed $\varnothing$-sequents of type V and Φ an S_n-signed $\varnothing$-sequent such that $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Delta \to \Phi)$ while $\mathcal{M}^{\mathfrak{A}} \not\models \Phi$. Then, there is some functional $\Psi \subseteq \Phi$ of type V such that $\Psi \in \Delta{\Uparrow}_{\preccurlyeq}$.*

COROLLARY 2.14. *Let $V \subseteq \mathrm{Var}$, Δ a finite set of functional S_n-signed $\varnothing$-sequents of type V forming an anti-chain with respect to $\preccurlyeq$ and Γ a finite set of S_n-signed $\varnothing$-sequents. Assume $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge \Delta \leftrightarrow \bigwedge \Gamma)$. Then, the following hold:*

(i) *There is an injection $f : \Delta \to \Gamma$ such that, for each $\Phi \in \Delta$, $|\Phi| \leqslant |f(\Phi)|$. In particular, $|\Delta| \leqslant |\Gamma|$.*
(ii) *If Γ consists of functional S_n-signed $\varnothing$-sequents of any type $V' \subseteq \mathrm{Var}$ and forms an anti-chain with respect to $\preccurlyeq$ then $\Gamma = \Delta$.*

PROOF: (i) Let Θ be the set of all those members of Γ which are not valid in $\mathcal{M}^{\mathfrak{A}}$. Then, by Corollary 2.13, for each $\Psi \in \Theta$, there are some functional $\Upsilon \subseteq \Psi$ and some $h(\Psi) \in \Delta$ such that $h(\Psi) \preccurlyeq \Upsilon$, in which case $\mathcal{M}^{\mathfrak{A}} \models (h(\Psi) \to \Psi)$. This yields a mapping $h : \Theta \to \Delta$ such that $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge h[\Theta] \to \bigwedge \Theta)$. Hence, $\mathcal{M}^{\mathfrak{A}} \models (\bigwedge h[\Theta] \to \bigwedge \Delta)$. Consider any $\Phi \in \Delta$. By Lemma 2.12, there is some $\Upsilon \in h[\Theta]$ such that $\Upsilon \preccurlyeq \Phi$. As Δ is an anti-chain with respect to $\preccurlyeq$, we get $\Upsilon = \Phi$. Hence, $\Phi \in h[\Theta]$. Thus, $h[\Theta] = \Delta$. Therefore, there is an injection f from Δ into Θ (and so, into Γ as well) such that, for each $\Phi \in \Delta$, $h(f(\Phi)) = \Phi$. Then, for each $\Phi \in \Delta$, there is some functional $\Psi \subseteq f(\Phi)$ such that $\Phi \preccurlyeq \Psi$, in which case $|\Phi| \leqslant |\Psi| \leqslant |f(\Phi)|$.

(ii) Now assume Γ consists of functional S_n-signed $\varnothing$-sequents of any type $V' \subseteq \mathrm{Var}$ and forms an anti-chain with respect to $\preccurlyeq$. Take any $\Phi \in \Delta$. By Corollary 2.13, there is then some $\Psi \in \Gamma$ such that $\Psi \preccurlyeq \Phi$. Hence, by the same corollary, there is some $\Upsilon \in \Delta$ such that $\Upsilon \preccurlyeq \Psi$, in which case $\Upsilon \preccurlyeq \Phi$. Since Δ forms an anti-chain with respect to $\preccurlyeq$, we get $\Upsilon = \Phi$, in which case we have both $\Phi \preccurlyeq \Psi$ and $\Psi \preccurlyeq \Phi$, so $\Psi = \Phi$. Thus, $\Phi \in \Gamma$. Therefore, $\Delta \subseteq \Gamma$. By symmetry, the converse inclusion equally holds. In this way, $\Gamma = \Delta$. □

In view of (1), Corollary 2.14 immediately yields

Theorem 2.15. *Let* $R = \dfrac{\Gamma}{\{\sigma{:}\lambda(p_l)_{l\in k}\}}$ *be an introduction rule for* $\mathcal{M}^{\mathfrak{A}}$. *Then, the following hold:*

(i) *There is an injection* $f : P^{\mathfrak{A}}_{\sigma:\lambda} \to \Gamma$ *such that, for each* $\Phi \in P^{\mathfrak{A}}_{\sigma:\lambda}$, $|\Phi| \leqslant |f(\Phi)|$. *In particular,* $|P^{\mathfrak{A}}_{\sigma:\lambda}| \leqslant |\Gamma|$.

(ii) *If* Γ *consists of functional* S_n*-signed* $\varnothing$*-sequents of any type* $V \subseteq \mathrm{Var}$ *and forms an anti-chain with respect to* $\preccurlyeq$ *then* $\Gamma = P^{\mathfrak{A}}_{\sigma:\lambda}$.

Remark 2.16. Corollary 2.9 together with Theorem 2.15(ii) yield the following procedure of constructing the set $P^{\mathfrak{A}}_{\sigma:\lambda}$ starting from a given introduction rule for $\mathcal{M}^{\mathfrak{A}}$ of type $\sigma{:}\lambda$. Let Γ be the set of all premises of such a rule. Put $\Delta \triangleq \{\Phi \in \Gamma \mid \mathcal{M}^{\mathfrak{A}} \not\models \Phi\}$. For each $\Phi \in \Delta$, put $\Phi{\downharpoonleft}N \triangleq \Phi \cap ((\{F_i \mid i \in n \setminus 1\}{:}\{p_l \mid l \in N\}) \cup (\{I_j \mid j \in n-1\}{:}\{p_m \mid m \in k \setminus N\}))$. Clearly, $\mathcal{M}^{\mathfrak{A}} \models (\Phi{\downharpoonleft}N \to \Phi)$. Moreover, by Corollary 2.9, $\mathcal{M}^{\mathfrak{A}} \models (\{\sigma{:}\lambda(p_l)_{l\in k}\} \to \Phi{\downharpoonleft}N)$. Note also that $\Phi{\downharpoonleft}N$ is of type $\{p_l \mid l \in N\}$. Hence, by Remark 2.6 and Theorem 2.15(ii), $P^{\mathfrak{A}}_{\sigma:\lambda} = \mathrm{Min}_{\preccurlyeq}\overrightarrow{\Delta{\downharpoonleft}N}$, where $\Delta{\downharpoonleft}N \triangleq \{\Phi{\downharpoonleft}N \mid \Phi \in \Delta\}$.

Remark 2.17. In view of Note 2.10 and Remark 2.16, we have the following effective procedure of automatic generation of the set $P^{\mathfrak{A}}_{\sigma:\lambda}$. For each $l \in N$, put $\Sigma^{n-1}_l \triangleq \varnothing$ and, for every $i \in n-1$, $\Sigma^i_l \triangleq \{F_{i+1}{:}p_l\}$. Likewise, for each $m \in n \setminus N$, set $\Sigma^0_m \triangleq \varnothing$ and, for every $j \in n \setminus 1$, $\Sigma^j_m \triangleq \{I_{j-1}{:}p_m\}$. Then, for any $\bar{a} \in n^k$, we have a functional S_n-signed $\varnothing$-sequent $\Sigma^{\bar{a}} \triangleq \bigcup_{l\in k} \Sigma^{a_l}_l$ of type $\{p_l\}_{l\in N}$. It is easy to see that $\overrightarrow{Q^{\mathfrak{A}}_{\sigma:\lambda}\lfloor N} = \{\Sigma^{\bar{a}} \mid \bar{a} \in n^k, \lambda^{\mathfrak{A}}(\bar{a}) \notin \sigma\}$. Define a partial ordering $\leqslant_N$ on n^k as follows. For all $\bar{a}, \bar{b} \in n^k$, put $\bar{a} \leqslant_N \bar{b}$ iff, for each $l \in N$, $a_l \geqslant b_l$ and, for every $m \in n \setminus N$, $a_m \leqslant b_m$. Clearly, $\Sigma^{\bar{a}} \preccurlyeq \Sigma^{\bar{b}}$ iff $\bar{a} \leqslant_N \bar{b}$. Hence, $P^{\mathfrak{A}}_{\sigma:\lambda} = \{\Sigma^{\bar{a}} \mid \bar{a} \in \mathrm{Min}_{\leqslant_N} (\lambda^{\mathfrak{A}})^{-1}[n \setminus \sigma]\}$.

Thus, Remark 2.17 yields a procedure of calculating $P^{\mathfrak{A}}_{\sigma:\lambda}$ of computational complexity n^{2k}. On the other hand, Remark 2.16 provides a quite useful tool of heuristic construction of an analytical expression of $P^{\mathfrak{A}}_{\sigma:\lambda}$ for all logics in denumerable families with unlimited number of truth values at once, as we demonstrate in the next section.

Next, we highlight certain general peculiarities of the issue involved.

Note 2.18. By functionality, every member of $P^{\mathfrak{A}}_{\sigma:\lambda}$ has at most k signed formulas.

The case $k = 0$ is analyzed in Remark 3.11 of [5]. Others are partially discussed below.

Remark 2.19. In case $k = 1$, $\lambda^{\mathfrak{A}}$ is either monotonic or anti-monotonic. Then, $(\lambda^{\mathfrak{A}})^{-1}[\sigma]$ is either a filter or an ideal. Hence, by Theorem 2.15(ii), $P^{\mathfrak{A}}_{\sigma:\lambda} = \varnothing$, if $(\lambda^{\mathfrak{A}})^{-1}[\sigma] = n$, $P^{\mathfrak{A}}_{\sigma:\lambda} = \{\varnothing\}$, if $(\lambda^{\mathfrak{A}})^{-1}[\sigma] = \varnothing$, and $P^{\mathfrak{A}}_{\sigma:\lambda} = \{\{(\lambda^{\mathfrak{A}})^{-1}[\sigma]{:}p_0\}\}$, elsewhere. If, in addition, $\lambda^{\mathfrak{A}}[\{0, n-1\}] = \{0, n-1\}$ then $\varnothing \neq (\lambda^{\mathfrak{A}})^{-1}[\sigma] \neq n$, in which case $P^{\mathfrak{A}}_{\sigma:\lambda} = \{\{(\lambda^{\mathfrak{A}})^{-1}[\sigma]{:}p_0\}\}$.[5]

Thus, when $k \leqslant 1$, $P^{\mathfrak{A}}_{\sigma:\lambda}$ has at most one element. Otherwise, the situation is more complicated.

Example 2.20. In case $k = 2$ and $\lambda^{\mathfrak{A}}$ is either min or max, by Theorem 2.15(ii), $P^{\mathfrak{A}}_{\sigma:\lambda} = \{\{\sigma{:}p_0\}, \{\sigma{:}p_1\}\}$, if either σ is a filter and $\lambda^{\mathfrak{A}} = \min$ or σ is an ideal and $\lambda^{\mathfrak{A}} = \max$, and $P^{\mathfrak{A}}_{\sigma:\lambda} = \{\{\sigma{:}p_0, \sigma{:}p_1\}\}$, otherwise.

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[5] Such is the case for the negation connectives of Dunn's, Gödel's and Łukasiewicz's logics.

In general, we have the following limitation on the power of $P^{\mathfrak{A}}_{\sigma:\lambda}$ in case $k \geqslant 2$.

Remark 2.21. Assume $k \geqslant 2$. Let Z be a set of functional subsets of $S_n{:}\{p_l\}_{l\in k}$ of any type $V \subseteq \{p_l\}_{l\in k}$ forming an anti-chain with respect to $\preccurlyeq$. Since both $\{F_i \mid i \in n \setminus 1\}$ and $\{I_j \mid j \in n-1\}$ are chains by inclusion, for any $\Phi, \Psi \in Z$ such that $\Phi \cap (S_n{:}\{p_l\}_{l\in k-1}) = \Psi \cap (S_n{:}\{p_l\}_{l\in k-1})$, either $\Phi \preccurlyeq \Psi$ or $\Psi \preccurlyeq \Phi$, in which case $\Phi = \Psi$. Therefore, $|Z| \leqslant n^{k-1}$.[6] In particular, $|P^{\mathfrak{A}}_{\sigma:\lambda}| \leqslant n^{k-1}$.[7]

Finally, consider corresponding calculi. Put $\mathcal{C}^{\mathfrak{A}} \triangleq \mathrm{Ax}_n \cup \{R^{\mathfrak{A}}_{s:\lambda} \mid s \in S_n, \lambda \in L\}$. Let $\overline{\mathcal{C}}^{\mathfrak{A}}$ be the S_n-sequent L-calculus obtained from $\mathcal{C}^{\mathfrak{A}}$ by adding all Weakenings and all $\{F_i, I_{i-1}\}$-Cuts, where $i \in n \setminus 1$. Then, by Theorem 2.11, Lemma 2.1, Remark 2.2, Corollary 2.4, [4] and [5], we have

Proposition 2.22 (Weak Completeness Theorem). *Any S_n-signed L-sequent is derivable in $\mathcal{C}^{\mathfrak{A}}$ iff it is valid in $\mathcal{M}^{\mathfrak{A}}$.*

Theorem 2.23 (Strong Completeness Theorem). *Any S_n-signed L-rule is derivable in $\overline{\mathcal{C}}^{\mathfrak{A}}$ iff it is valid in $\mathcal{M}^{\mathfrak{A}}$.*

As an immediate consequence of Proposition 2.22 and Theorem 2.23, we obtain

Corollary 2.24 (Cut Elimination Theorem). *$\overline{\mathcal{C}}^{\mathfrak{A}}$ has the cut-elimination property.*

3. Applications

As for Dunn's, Łukasiewicz's and Gödel's finitely-valued logics, in view of Remark 2.19 and Example 2.20, it only remains to specify rules of introduction of negation and to find rules of introduction of implication. This task is solved below for Dunn's and Gödel's logics. Concerning Łukasiewicz's

[6] For this reason, when $k \geqslant 2$, the computational complexity of the effective procedure resulted from Remark 2.17 is rather n^{2k-1} than n^{2k}.

[7] As it is shown in Subsection 3.2, this limit is reachable, at least, if $k = 2$.

logics, the problem involved has been resolved directly and is going to be presented elsewhere.

3.1. Dunn's finitely-valued logics

The implication in Dunn's logics is defined as follows:

$$a \supset^{\mathfrak{A}} b \triangleq \begin{cases} \max(n-1-a, b) & \text{if } a \leqslant b, \\ \min(n-1-a, b) & \text{otherwise,} \end{cases} \tag{1}$$

for all $a, b \in n$, while the negation is given by $\neg^{\mathfrak{A}} c \triangleq n-1-c$, for all $c \in n$. Clearly, by Remark 2.19, for each $i \in n \setminus 1$ and every $j \in n-1$, $P^{\mathfrak{A}}_{F_i:\neg} = \{\{I_{n-1-i}{:}p_0\}\}$ and $P^{\mathfrak{A}}_{I_j:\neg} = \{\{F_{n-1-j}{:}p_0\}\}$.

THEOREM 3.1. *For each $i \in n \setminus 1$ and every $j \in n-1$:*

$$\begin{aligned} P^{\mathfrak{A}}_{F_i:\supset} &= \{\{I_{n-1-i}{:}p_0, F_i{:}p_1\} \mid 2i > n-1\} \cup \\ &\quad \{\{I_{l-1}{:}p_0, F_l{:}p_1\} \mid l \in n \setminus 1, \\ &\quad (l \geqslant \max(n-i, i+1) \textit{ or } l \leqslant \min(i, n-i-1))\}, \\ P^{\mathfrak{A}}_{I_j:\supset} &= \{\{F_{n-1-j}{:}p_0, I_j{:}p_1\} \mid 2j < n-2\} \cup \{\{F_1{:}p_0\}\} \cup \\ &\quad \{\{F_m{:}p_0, I_{m-2}{:}p_1\} \mid m \in n \setminus 2, \\ &\quad (m \leqslant \min(n-1-j, j+1) \textit{ or } m \geqslant \max(j+2, n-j))\} \end{aligned}$$

PROOF: Since, for all $a, b \in n$, $\min(a, b) \leqslant \max(a, b)$, $a \leqslant b \Leftrightarrow \forall l \in n \setminus 1 : a \in F_l \Rightarrow b \in F_l$ and $a > b \Leftrightarrow b+1 \leqslant a \Leftrightarrow a \in F_1$ and $\forall m \in n \setminus 2 : b \in F_{m-1} \Rightarrow a \in F_m$, (1) immediately implies the fact that $X \triangleq \{\{I_{n-1-i}{:}p_0, F_i{:}p_1\}\} \cup \{\{I_{l-1}{:}p_0, F_l{:}p_1, I_{n-1-i}{:}p_0\}, \{I_{l-1}{:}p_0, F_l{:}p_1, F_i{:}p_1\} \mid l \in n \setminus 1\}$ and $Y \triangleq \{\{F_{n-1-j}{:}p_0, I_j{:}p_1\}, \{F_1{:}p_0, F_{n-1-j}{:}p_0\}, \{F_1{:}p_0, I_j{:}p_1\}\} \cup \{\{F_m{:}p_0, I_{m-2}{:}p_1, F_{n-1-j}{:}p_0\}, \{F_m{:}p_0, I_{m-2}{:}p_1, I_j{:}p_1\} \mid m \in n \setminus 2\}$ are the sets of premises of introduction rules for $\mathcal{M}^{\mathfrak{A}}$ of types $F_i{:}\supset$ and $I_j{:}\supset$, respectively. Notice that all members of X are of type $\{p_1\}$ whereas those of Y are of type $\{p_0\}$. Then, $\overrightarrow{X} = \{\{I_{n-1-i}{:}p_0, F_i{:}p_1\}\} \cup \{\{I_{l-1}{:}p_0, F_l{:}p_1\} \mid l \in n \setminus 1, (l \geqslant n-i \text{ or } l \leqslant i)\} \cup \{\{F_l{:}p_1, I_{n-1-i}{:}p_0\} \mid l \in n \setminus 1, l < n-i\} \cup \{\{I_{l-1}{:}p_0, F_i{:}p_1\} \mid l \in n \setminus 1, l > i\}$ and $\overrightarrow{Y} = \{\{F_{n-1-j}{:}p_0, I_j{:}p_1\}, \{F_1{:}p_0\}, \{F_1{:}p_0, I_j{:}p_1\}\} \cup \{\{F_m{:}p_0, I_{m-2}{:}p_1\} \mid m \in n \setminus 2, (m \leqslant n-1-j \text{ or } m \geqslant j+2)\} \cup \{\{I_{m-2}{:}p_1, F_{n-1-j}{:}p_0\} \mid m \in n \setminus 2, m > n-1-j\} \cup \{\{F_m{:}p_0, I_j{:}p_1\} \mid m \in n \setminus 2, m < j+2\}$. Remark that all members of $\{\{F_l{:}p_1, I_{n-1-i}{:}p_0\} \mid l \in$

$n \setminus 1, l < n-i\}$ and $\{\{I_{l-1}{:}p_0, F_i{:}p_1\} \mid l \in n \setminus 1, l > i\}$ are subsumed by the sequents $\{I_{n-1-i}{:}p_0, F_{n-i}{:}p_1\}$ and $\{I_{i-1}{:}p_0, F_i{:}p_1\}$, respectively, which belong to the rest of $\overrightarrow{X}$. Likewise, all elements of $\{\{I_{m-2}{:}p_1, F_{n-1-j}{:}p_0\} \mid m \in n \setminus 2, m > n-1-j\}$ and $\{\{F_m{:}p_0, I_j{:}p_1\} \mid m \in n \setminus 2, m < j+2\}$ are subsumed by the sequents $\{F_{n-1-j}{:}p_0, I_{n-3-j}{:}p_1\}$ and $\{F_{j+2}{:}p_0, I_j{:}p_1\}$, respectively, which belong to the rest of $\overrightarrow{Y}$. Moreover, $\{F_1{:}p_0\} \preccurlyeq \{F_1{:}p_0, I_j{:}p_1\}$. It is easy to see that both $\{\{I_{l-1}{:}p_0, F_l{:}p_1\} \mid l \in n \setminus 1, (l \geqslant n-i \text{ or } l \leqslant i)\}$ and $\{\{F_1{:}p_0\}\} \cup \{\{F_m{:}p_0, I_{m-2}{:}p_1\} \mid m \in n \setminus 2, (m \leqslant n-1-j \text{ or } m \geqslant j+2)\}$ are anti-chains with respect to $\preccurlyeq$. Further, for any $l \in n \setminus 1$, $\{I_{n-1-i}{:}p_0, F_i{:}p_1\} \preccurlyeq \{I_{l-1}{:}p_0, F_l{:}p_1\}$ iff $n-i \leqslant l \leqslant i$. Likewise, for any $m \in n \setminus 2$, $\{F_{n-1-j}{:}p_0, I_j{:}p_1\} \preccurlyeq \{F_m{:}p_0, I_{m-2}{:}p_1\}$ iff $j+2 \leqslant m \leqslant n-1-j$. Moreover, $\{F_{n-1-j}{:}p_0, I_j{:}p_1\} \not\preccurlyeq \{F_1{:}p_0\}$. Finally, assume there is some $l \in n \setminus 1$ such that $\{I_{l-1}{:}p_0, F_l{:}p_1\} \preccurlyeq \{I_{n-1-i}{:}p_0, F_i{:}p_1\}$ and either $l \geqslant \max(n-i, i+1)$ or $l \leqslant \min(i, n-i-1)$. Then, $i \leqslant l \leqslant n-i$, in which case $2i \leqslant n-1$. Conversely, assume $2i \leqslant n-1$. Then, $n-i = \max(n-i, i+1)$. Moreover, $n-i > i$, so $\{I_{n-1-i}{:}p_0, F_{n-i}{:}p_1\} \preccurlyeq \{I_{n-1-i}{:}p_0, F_i{:}p_1\}$. Likewise, suppose there is some $m \in n \setminus 2$ such that $\{F_m{:}p_0, I_{m-2}{:}p_1\} \preccurlyeq \{F_{n-1-j}{:}p_0, I_j{:}p_1\}$ and either $m \leqslant \min(n-1-j, j+1)$ or $m \geqslant \max(j+2, n-j)$. Then, $n-1-j \leqslant m \leqslant j+2$, in which case $2j \geqslant n-2$. Assume $\{F_1{:}p_0\} \preccurlyeq \{F_{n-1-j}{:}p_0, I_j{:}p_1\}$. Then, $n-1-j = 1$, that is, $j = n-2$. Since $n \geqslant 2$, $2j \geqslant j = n-2$ as well. Conversely, suppose $2j \geqslant n-2$. Then, $j+2 = \max(j+2, n-j)$. Moreover, $j+2 > n-1-j$, so $\{F_{j+2}{:}p_0, I_j{:}p_1\} \preccurlyeq \{F_{n-1-j}{:}p_0, I_j{:}p_1\}$. In this way, Remark 2.16 completes the argument. □

3.2. Gödel's finitely-valued logics

The implication in Gödel's logics is defined as follows:

$$a \supset^{\mathfrak{A}} b \triangleq \begin{cases} n-1 & \text{if } a \leqslant b, \\ b & \text{otherwise,} \end{cases}$$

for all $a, b \in n$, whereas the negation is given by $\neg^{\mathfrak{A}} c \triangleq c \supset^{\mathfrak{A}} 0$, for all $c \in n$. Clearly, by Remark 2.19, for each $s \in S_n$, $P^{\mathfrak{A}}_{s:\neg} = \{\{I_0{:}p_0\}\}$, if s is a filter, and $P^{\mathfrak{A}}_{s:\neg} = \{\{F_1{:}p_0\}\}$, otherwise.

THEOREM 3.2. *For each $i \in n \setminus 1$ and every $j \in n-1$:*

$$\begin{aligned} P^{\mathfrak{A}}_{F_i:\supset} &= \{\{I_{l-1}{:}p_0, F_l{:}p_1\} \mid l \in n \setminus 1, l \leqslant i\}, \\ P^{\mathfrak{A}}_{I_j:\supset} &= \{\{I_j{:}p_1\}, \{F_1{:}p_0\}\} \\ &\cup \{\{F_m{:}p_0, I_{m-2}{:}p_1\} \mid m \in n \setminus 2, m \leqslant j+1\}. \end{aligned}$$

PROOF: Notice that, for all $a, b \in n$, $a \supset^{\mathfrak{A}} b \in F_i$ iff either $a \leqslant b$ or $b \in F_i$, whereas $a \supset^{\mathfrak{A}} b \in I_j$ iff both $a > b$ and $b \in I_j$. Then, taking the first sentence of the proof of Theorem 3.1 into account, we conclude that $X \triangleq \{\{I_{l-1}{:}p_0, F_l{:}p_1, F_i{:}p_1\} \mid l \in n \setminus 1\}$ and $Y \triangleq \{\{I_j{:}p_1\}, \{F_1{:}p_0\}\} \cup \{\{F_m{:}p_0, I_{m-2}{:}p_1\} \mid m \in n \setminus 2\}$ are the sets of premises of introduction rules for $\mathcal{M}^{\mathfrak{A}}$ of types $F_i{:}\supset$ and $I_j{:}\supset$, respectively. Remark that Y consists of functional S_n-signed $\varnothing$-sequents of type $\{p_0\}$. Moreover, $Y \setminus \{\{I_j{:}p_1\}\}$ is an anti-chain with respect to $\preccurlyeq$, no member of which subsumes $\{I_j{:}p_1\}$. Clearly, $\{I_j{:}p_1\} \not\preccurlyeq \{F_1{:}p_0\}$. Furthermore, for each $m \in n \setminus 2$, $\{I_j{:}p_1\} \preccurlyeq \{F_m{:}p_0, I_{m-2}{:}p_1\}$ iff $m \geqslant j+2$. As for the set X, it consists of S_n-signed $\varnothing$-sequents of type $\{p_1\}$. Then, $\overrightarrow{X} = \{\{I_{l-1}{:}p_0, F_l{:}p_1\} \mid l \in n \setminus 1, l \leqslant i\} \cup \{\{I_{l-1}{:}p_0, F_i{:}p_1\} \mid l \in n \setminus 1, l > i\}$. Clearly, every member of $\{\{I_{l-1}{:}p_0, F_i{:}p_1\} \mid l \in n \setminus 1, l > i\}$ is subsumed by the sequent $\{I_{i-1}{:}p_0, F_i{:}p_1\}$ which belongs to the rest of $\overrightarrow{X}$ which, in its turn, is an anti-chain with respect to $\preccurlyeq$. Then, Remark 2.16 completes the argument. □

Notice that $P^{\mathfrak{A}}_{I_{n-2}:\supset}$ has exactly n elements.

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