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M. Łazarz, K. Siemieńczuk

NEW CHARACTERIZATION OF SKELETON OF FINITE DISTRIBUTIVE LATTICE

Abstract

In the paper we introduce the notion of $\to$-irreducibility and we show that the set of all $\to$-irreducible elements of a finite Heyting lattice L forms the skeleton of L. We also discuss a parallel concept of $\leftrightarrow$-irreducibility and give a similar characterization. Finally, we present generalizations of these results for some class of infinite Heyting lattices.
Keywords: skeleton, distributive lattice, finite length, $\to$-irreducible element.

1. Basic concepts and motivation

In this section we introduce standard lattice-theoretic concepts and facts, that can be found in [1] and [7]. For more technical notions connected with theory of tolerance relations see [6].

A lattice L is *distributive* if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$. At least two-element distributive lattice with the least element 0 and the greatest element 1 is a *Boolean lattice*, if every element has a *complement*, i.e., for every x there is y such that $x \wedge y = 0$ and $x \vee y = 1$.

For any L and $x, y \in L$ such that $x \leq y$, the set $[x, y] = \{z \in L : x \leq z \leq y\}$ is called an *interval* of L. If $[x, y]$ consists of exactly two elements then we say that y *covers* x and write $x \prec y$.

In the paper we use two well known facts which are true in the domain of distributive lattices of a finite length. First, so called the Isomorphism Theorem (see [7], p. 162): for any x and y intervals $[x \wedge y, x]$, $[y, x \vee y]$ are isomorphic. Second, Theorem VII.6 from [2]: a lattice is complemented iff 1 is the supremum of atoms (i.e. covers of 0).

An interval $[x, y]$ is a *Boolean interval* if $[x, y]$ is Boolean as a lattice. A Boolean interval $[x, y]$ is said to be

- an MBI (an abbreviation of *maximal Boolean interval*) iff for every Boolean interval $[z, t]$ if $[x, y] \subseteq [z, t]$ then $[x, y] = [z, t]$;
- an MBI* iff for every Boolean interval $[x, t]$ if $[x, y] \subseteq [x, t]$ then $[x, y] = [x, t]$;
- an MBI$_*$ iff for every Boolean interval $[z, y]$ if $[x, y] \subseteq [z, y]$ then $[x, y] = [z, y]$.

It is easy to see that an interval is an MBI iff it is an MBI* and an MBI$_*$. If L is finite, then for $x < 1$ we define $x^* = \sup\{y \in L : x \prec y\}$ and $1^* = 1$. Similarly for $x > 0$ let $x_* = \inf\{y \in L : y \prec x\}$ and $0_* = 0$. In the case when L is distributive, $[x, x^*]$ is an MBI* if $x < 1$, and $[x_*, x]$ is an MBI$_*$ if $x > 0$ (it is a consequence of Theorem VII.6 from [2]).

A distributive lattice L is said to be a *Heyting lattice*, if for any $x, y \in L$ there is the greatest element u such that $x \wedge u \leq y$; in that case this element is denoted by $x \to y$ (so called *pseudocomplement of* x *relative to* y). In other words, L is a Heyting lattice if for any $x, y \in L$ there is some element $x \to y \in L$ such that $u \leq x \to y$ iff $x \wedge u \leq y$. It is easy to see that the class of finite distributive lattices and the class of finite Heyting lattices coincide.

Let L be a finite distributive lattice and R a binary relation on L. If R is reflexive and symmetric and preserves lattice operations (i.e., xRy and zRt imply $(x \vee z)R(y \vee t)$ and $(x \wedge z)R(y \wedge t)$) then it is called a *tolerance relation* (see [3]). A set $B \subseteq L$ is said to be a *block* (of tolerance R) if it is a maximal subset of L (in the sense of inclusion) such that xRy for any $x, y \in B$. Two blocks need not to be disjoint, but every block B is an interval of L, so it has the least element 0_B and the greatest element 1_B, and hence $B = [0_B, 1_B]$. Let L/R stand for the family of all blocks. The set L/R with an order defined as follows:

$$A \leq B \;\Leftrightarrow\; 0_A \leq 0_B$$

forms a lattice (so called *factor lattice*, see [4]). On the other hand, the set $\{0_B \in L : B \in L/R\}$ is not in general a sublattice of L, although it is closed under suprema.

A tolerance relation R on L is said to be *glued* iff $A \prec B$ implies $A \cap B \neq \emptyset$, for any $A, B \in L/R$. The intersection of all glued tolerance relations on L is the smallest glued tolerance relation on L (see [5], p. 315) and it is denoted by Σ. Let us recall the central notion considered in the paper (introduced by Ch. Herrmann in [8]): the *skeleton* of a lattice L, denoted by $S(L)$, is the factor lattice L/Σ. Since L/Σ and $\{0_B \in L : B \in L/\Sigma\}$ are isomorphic (with respect to order) we may assume that $S(L) = \{0_B \in L : B \in L/\Sigma\}$.

In the case of finite distributive lattice L it has been proved (see [9]) that blocks of Σ are maximal Boolean intervals of L, so the skeleton of L can be presented in the following form:

$$S(L) = \{x \in L : [x, y] \textit{ is an MBI, for some } y \in L\}.$$

Since finite distributive lattices are Heyting lattices, the natural question arises: how to describe the notion of skeleton of a finite distributive lattice in terms of $\to$? This was a motivation of our work.

2. Characterization of the notion of skeleton by means of $\to$-irreducible elements

Let L be a Heyting lattice. An element $a \in L$ is $\to$-irreducible iff $a = x \to y$ implies $a = x$ or $a = y$, for any $x, y \in L$. It is easy to see that if we assume that $a = x \to y$ then disjunction $a = x$ or $a = y$ is equivalent to $a = y$. Hence the notion of $\to$-irreducibility we define as follows:

Definition 1. *An element a of a Heyting lattice L is said to be $\to$-irreducible if $a = x \to y$ implies $a = y$, for any $x, y \in L$.*

The following lemma is a straightforward consequence of Theorem VII.6 from [2].

Lemma 1. *Let L be a distributive lattice and $x, y, z \in L$. Then*

1. *if $[x, y]$ and $[x, z]$ are Boolean intervals then $[x, y \vee z]$ is also Boolean,*
2. *if $[x, z]$ and $[y, z]$ are Boolean intervals then $[x \wedge y, z]$ is also Boolean.*

The converse of Lemma 1 need not be true since conditions (1) and (2) of Lemma 1 may be trivially fulfilled: consider a circle on the plane as a lattice with a dense order. However, it can be shown that if lattice L

is of a finite length, then conditions (1) and (2) imply distributivity of L (see [10]). Note also that (1) and (2) of Lemma 1 are equivalent in modular lattices.

The main result of the paper is the following theorem, providing a characterization of the skeleton of a finite distributive lattice in terms of $\to$-irreducible elements.

THEOREM 1. *In a finite Heyting lattice, an element a is $\to$-irreducible iff $[a, a^*]$ is an MBI.*

COROLLARY 1. *The skeleton of a finite distributive lattice L can be characterized as follows:*

$$S(L) = \{x \in L : x \textit{ is } \to\textit{-irreducible}\}.$$

LEMMA 2. *If $[a, a^*]$ is not an MBI, then a is $\to$-reducible.*

PROOF. Since $[a, a^*]$ is not an MBI, then there is some $b < a$ such that $[b, a^*]$ is a Boolean interval. Let a' stand for the complement of a in $[b, a^*]$. Consider $u = a' \to b$; we will show that $u = a$.

Obviously $a \leq u$, since $a' \wedge a \leq b$. Moreover $a' \wedge u \wedge a^* = b$ so adding a to both sides, we obtain $a^* \wedge u = a$. Suppose that $a < u$ and take v such that $a \prec v \leq u$. By definition of a^* we obtain $v \leq a^*$ so $a \prec v \leq a^* \wedge u = a$; a contradiction. ■

LEMMA 3. *If $[a, a^*]$ is an MBI, then a is $\to$-irreducible.*

PROOF. Suppose that $a = x \to y$ and $y < a$. First of all, observe that:

$$a^* \leq x \vee a.$$

If not, there is some z such that $a \prec z$ and $z \not\leq x \vee a$. Since $a \leq (x \vee a) \wedge z \leq z$, we get $(x \vee a) \wedge z = a$, so adding x to both sides we obtain

$$x \wedge z = x \wedge (x \vee a) \wedge z = x \wedge a \leq y$$

and hence $z \leq x \to y = a$; a contradiction.

Now, choose an element v such that $y \leq v \prec a$. By proven inequality and distributivity it is easy to show that the set $\{v, (x \wedge a^*) \vee v, a, a^*\}$ forms a sublattice, so by Isomorphism Theorem we obtain $(x \wedge a^*) \vee v \prec a^*$. Finally, by Lemma 1 interval $[v, a^*]$ is Boolean; a contradiction. ■

COROLLARY 2. *Let L be a finite Heyting lattice. For every $a \in L$ either a is $\rightarrow$-irreducible or $a = x \rightarrow y$ where y is $\rightarrow$-irreducible.*

PROOF. Let a be $\rightarrow$-reducible. Consider an MBI* $[(a^*)_*, ((a^*)_*)^*]$. In fact by Lemma 1, this interval is an MBI and a belongs to it. Therefore, by Lemma 3 the element $(a^*)_*$ is $\rightarrow$-irreducible. Now, mimicking the proof of Lemma 2 we show that $a = a' \rightarrow (a^*)_*$, where a' is the complement of a in $[(a^*)_*, ((a^*)_*)^*]$. ■

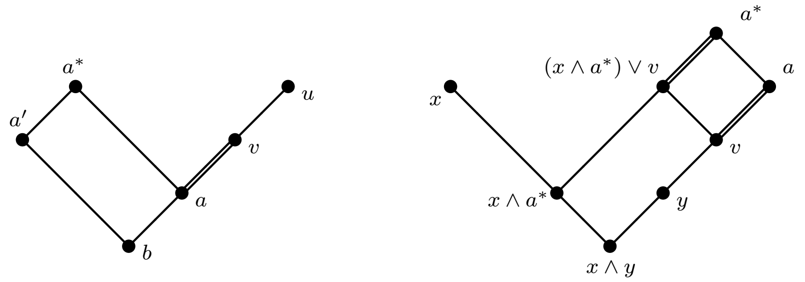

Figure 1: The idea of proofs of Lemmas 2 and 3 (double lines denote covering relations).

3. $\leftrightarrow$-irreducibility

If x and y are arbitrary elements of a Heyting lattice L then symbol $x \leftrightarrow y$ denotes $(x \rightarrow y) \wedge (y \rightarrow x)$. Similarly to Section 2 we say that an element $a \in L$ is *$\leftrightarrow$-irreducible* if $a = x \leftrightarrow y$ implies $a = x$ or $a = y$, for any $x, y \in L$. As an application of Lemmas 2 and 3 we achieve a characterization of $\leftrightarrow$-irreducible elements.

THEOREM 2. *For any finite Heyting lattice L and $a \in L$ the following conditions are equivalent:*

1. *a is $\leftrightarrow$-irreducible,*
2. *a is $\rightarrow$ and $\wedge$-irreducible,*
3. *$[a, a^*]$ is a two-element MBI.*

Proof. (1) $\Rightarrow$ (2) This implication is clear by definitions and simple facts true in Heyting lattices: (a) if a is $\to$-reducible then $a = x \to y$ and $y < a$ and hence $a = x \leftrightarrow (x \wedge y)$, $a \neq x$ and $a \neq x \wedge y$, so a is $\leftrightarrow$-reducible. (b) On the other hand, if a is $\wedge$-reducible, say $a = x \wedge y$ where $x, y > a$, then we easily compute that $a = x \leftrightarrow (x \to a)$ and obviously $a \neq x$ and $a \neq x \to a$, so a is $\leftrightarrow$-reducible.

(2) $\Rightarrow$ (1) is obvious by definition of $\leftrightarrow$.

(2) $\Rightarrow$ (3) Since a is $\to$-irreducible then by Lemma 2, interval $[a, a^*]$ is an MBI. Hence $a < 1$, so there is b such that $a \prec b$. But since a is $\wedge$-irreducible $b = a^*$, so the interval $[a, a^*]$ has exactly two elements.

(3) $\Rightarrow$ (2) Since a^* is the only cover of a, the element a is $\wedge$-irreducible. Moreover, since $[a, a^*]$ is an MBI, then by Lemma 3, the element a is $\to$-irreducible. ■

4. Infinite Heyting lattices

It is easy to see that results of two previous sections may be generalized to some class of infinite Heyting lattices. In fact the assumption "L is finite" could be replaced by the conjunction of the following conditions:

($F1$) If $a < b$ then there is c, such that $a \prec c \leq b$.

($F2$) If $b < a$ then there is c, such that $b \leq c \prec a$.

($F3$) If $a < 1$ then there is an MBI* $[a, b]$.

($F4$) If $0 < a$ then there is an MBI$_*$ $[b, a]$.

Strictly speaking, taking any MBI* $[a, b]$ and MBI$_*$ $[c, a]$ instead of $[a, a^*]$ and $[a_*, a]$, respectively, one can repeat previous arguments and prove:

Corollary 3. *If a Heyting lattice L satisfies ($F1$)–($F3$) then for every $a \in L$, a is $\to$-irreducible iff a is the least element in some MBI.*

Corollary 4. *If a Heyting lattice L satisfies ($F1$)–($F4$) then for every $a \in L$, either a is $\to$-irreducible or $a = x \to y$, where y is $\to$-irreducible.*

Corollary 5. *If a Heyting lattice L satisfies ($F1$)–($F3$) then for every $a \in L$, a is $\leftrightarrow$-irreducible iff a is the least element in some two-element MBI.*

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Department of Logic and Methodology of Sciences
University of Wrocław, Poland
e-mail: lazarzmarcin@poczta.onet.pl
e-mail: logika6@gmail.com