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MODELS OF CUBIC THEORIES*

Abstract

Cubic structures and cubic theories are defined on a base of multidimensional cubes. Spectra and structures of models for cubic theories are described. It is proved that there exists an ω-stable locally ω-categorical cubic theory with colored edges such that the types of elements are defined by dimensions of comprehensive cubes, i.e., by eccentricities of elements, and each vertex of unique non-isolated 1-type is incident with infinitely many edges for each of countably many colors.

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In the article, we define theories of cubic structures and describe the spectra and the structure of models for cubic theories. An interest to cubic structures and their theories is explained by two circumstances. On the one hand, multidimensional cubes have sufficiently wide applications in Discrete Mathematics [1]. On the other hand, taking multidimensional cubes with colored edges it is convenient to model theories with various model-theoretic properties such as ω_1-categorical non-ω-categorical theories [2] and *locally ω-categorical theories* [3] for which dimensions of cubes define orbits of automorphism groups: any two elements have the same type if and only if these elements belong to cubes of the same finite dimension or dimensions of that cubes are infinite.

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The description of spectra for cubic theories partially corresponds to the description of spectra for theories of everywhere finitely defined polygonometries [6]. This link is explained by the property that cubes are graphs of polygonometries [7].

In Section 1, necessary definitions and preliminary assertions are introduced. The description of models for cubic theories of finite diameters is given in Section 2. In Sections 3 and 4, countable and, respectively, uncountable models of cubic theories are described. In Section 5, cubic theories with special colorings of edges are defined and, on a base of that theories, it is shown that there exists an ω-stable locally ω-categorical cubic theory with colored edges such that the types of elements are defined by dimensions of comprehensive cubes, i.e., by eccentricities of elements, and each vertex of unique non-isolated 1-type is incident with infinitely many edges for each of countably many colors.

Below we use, without specifications, the standard model-theoretic and graph-theoretic terminology [1], [4], [5].

1. Cubes and cubic theories

Recall [1] that a *n-dimensional cube*, or a *n-cube* (where $n \in \omega$) is a graph isomorphic to the graph $\mathcal{Q}_n$ with the universe $\{0,1\}^n$ and such that any two vertices $(\delta_1, \ldots, \delta_n)$ and $(\delta'_1, \ldots, \delta'_n)$ are adjacent if and only if these vertices differ exactly in one coordinate. The described graph $\mathcal{Q}_n$ is called the *canonical representative* for the class of n-cubes.

Let λ be an infinite cardinal. A *λ-dimensional cube*, or a *λ-cube*, is a graph isomorphic to a graph $\Gamma = \langle X; R\rangle$ satisfying the following conditions:

(1) the universe $X \subseteq \{0,1\}^\lambda$ is generated from an arbitrary function $f \in X$ by the operator $\langle f\rangle$ attaching, to the set $\{f\}$, all results of substitutions for any finite tuples $(f(i_1), \ldots, f(i_m))$ by tuples $(1-f(i_1), \ldots, 1-f(i_m))$;

(2) the relation R consists of edges connecting functions differing exactly in one coordinate[1].

The described graph $\mathcal{Q} \rightleftharpoons \mathcal{Q}_f$ with the universe $\langle f\rangle$ is a *canonical representative* for the class of λ-cubes.

Note that the canonical representative of the class of n-cubes (as well as the canonical representatives of the class of λ-cubes) are generated by

[1]Here and below, the *(i-th) coordinate* of function $g \in \{0,1\}^\lambda$ is the value $g(i)$ correspondent to the argument $i < \lambda$.

any its function: $\{0,1\}^n = \langle f\rangle$, where $f \in \{0,1\}^n$. Therefore the universes of canonical representatives $\mathcal{Q}_f$ of n-cubes like λ-cubes, will be denoted by $\langle f\rangle$.

We say that two functions $f_1, f_2 \in \{0,1\}^\lambda$ (where λ is a finite or an infinite cardinal) are *equivalent* and write $f_1 \sim f_2$ if $f_1 \in \langle f_2\rangle$.

Clearly, $\sim$ is an equivalence relation on any set $\{0,1\}^\lambda$. The relation $\sim$ on $\{0,1\}^\lambda$ has a unique equivalence class if λ is finite, and 2^λ equivalence classes if λ is infinite.

Any λ-cube $\mathcal{Q}$ is called a *multidimensional cube*, or simply a *cube*, and λ is a *dimension* of λ-cube $\mathcal{Q}$.

Note the following immediate properties of λ-cubes.

1.1. Each vertex of a λ-cube is incident to λ edges.

1.2. Each cube is connected.

1.3. For any cardinal λ all λ-cubes are pairwise isomorphic and have the cardinality 2^λ (for finite λ) or λ (for infinite λ).

1.4. Each finite subgraph Γ of a cube $\mathcal{Q}$ is extensible to the least, by inclusion, finite subgraph $\overline{\Gamma}$ in $\mathcal{Q}$ being a n-cube for some $n \in \omega$.

1.5. The cardinality 2^n of n-cube $\overline{\Gamma}$ is bounded by a function dependent only on cardinality of Γ and values for distances between vertices in Γ.

Let Γ be a subgraph of a cube $\mathcal{Q}_f$. Consider the operator FC, which transforms the subgraph Γ to a subgraph $\mathrm{FC}(\Gamma)$ obtained by adding to Γ all least (finite) cubes in $\mathcal{Q}_f$ containing arbitrary finite sets of vertices in Γ. By 1.4, the graph $\mathrm{FC}(\Gamma)$ exists and it is connected. Note that $\mathrm{FC}(\mathrm{FC}(\Gamma)) = \mathrm{FC}(\Gamma)$.

1.6. Each subgraph Γ of a cube $\mathcal{Q}$ and having an infinite cardinality μ is extensible to the least subgraph $\overline{\Gamma}$ in $\mathcal{Q}$ and being a μ-cube.

Below the least cube $\overline{\Gamma}$ in the cube $\mathcal{Q}$ will be denoted by $\mathrm{FC}_{\mathcal{Q}}(\Gamma)$ or simply by $\mathrm{FC}(\Gamma)$ if the cube $\mathcal{Q}$ is fixed.

1.7. A subgraph Γ of a cube $\mathcal{Q}$ is again a cube if and only if Γ, along with any two its vertices a and b, contains all vertices of all shortest (a,b)-routes in $\mathcal{Q}$.

Recall that for a set A in a theory T the *algebraic* (respectively *definable*) *closure* of A is the union of sets for solutions of formulas $\theta(x,\bar{a})$,

$\bar{a} \in A$, such that $\models \exists^{=n} x\, \theta(x, \bar{a})$ for some $n \in \omega$ ($\models \exists^{=1} x\, \theta(x, \bar{a})$). The algebraic closure of A is denoted by $\mathrm{acl}(A)$ and the definable closure by $\mathrm{dcl}(A)$.

1.8. Since between any two vertices a and b in a cube $\mathcal{Q}$ there are a finite and uniformly bounded (as pointed out in 1.5) number of shortest (a, b)-routes, for any subgraph $\Gamma = \langle X; R\rangle$ of $\mathcal{Q}$ the set of all vertices in $\mathrm{FC}_{\mathcal{Q}}(\Gamma)$ is contained in $\mathrm{acl}(X)$.

1.9. If $\mathcal{Q} = \langle X; R\rangle$ is a cube, a is a vertex in $\mathcal{Q}$, and A is the set of all vertices in $\mathcal{Q}$ adjacent with a then $X = \mathrm{dcl}(\{a\} \cup A)$.

Proof. Since any cube is connected it suffices to prove that each vertex b, being at distance 2 from a in the cube $\mathcal{Q}$, belongs to $\mathrm{dcl}(\{a\} \cup A)$. Indeed, if $\mathrm{dcl}(\{a\} \cup A)$ contains all vertices in $\mathcal{Q}$, being at a positive distance $\leq n$ from a, then any vertex c in $\mathcal{Q}$, being at the distance $n + 1$ from a, is situated at the distance 2 from a vertex d that already belongs to $\mathrm{dcl}(\{a\} \cup A)$ and has the distance $\rho(a, d) = n - 1$, moreover, all vertices adjacent to d are at a distance $\leq n$ from a. Then, by induction, we obtain $X = \mathrm{dcl}(\{a\} \cup A)$. We have the required property since there are exactly two vertices e_1 and e_2 in A adjacent with b and these vertices are adjacent simultaneously only with a and b. Thus, the formula $\theta(x, a, e_1, e_2) \rightleftharpoons R(x, e_1) \wedge R(x, e_2) \wedge \neg(x \approx a)$ witnesses that $b \in \mathrm{dcl}(\{a\} \cup A)$. $\square$

1.10. Any two μ-cubes, $\mu < \lambda$, lying in a λ-cube $\mathcal{Q}$ are linked by an automorphism in $\mathrm{Aut}(\mathcal{Q})$.

1.11. For any $\lambda \geq \omega$ any λ-cube is a λ-homogeneous structure.

Any graph $\Gamma = \langle X; R\rangle$, where any connected component is a cube, is called a *cubic structure*. A theory T of graph language $\{R^{(2)}\}$ is *cubic* if $T = \mathrm{Th}(\mathcal{M})$ for some cubic structure $\mathcal{M}$. In this case, the structure $\mathcal{M}$ is called a *cubic model* of T.

Below we consider only cubic theories T having infinite models. As usual we denote by $I(T, \lambda)$ the number of pairwise non-isomorphic models of T having the cardinality λ.

The *invariant* of theory T is the function

$$\mathrm{Inv}_T \colon \omega \cup \{\infty\} \to \omega \cup \{\infty\},$$

satisfying the following conditions:

(1) for any natural n, $\mathrm{Inv}_T(n)$ is the number of connected components in any model of T, being n-cubes, if that number is finite, and $\mathrm{Inv}_T(n) = \infty$ if that number is infinite;

(2) $\mathrm{Inv}_T(\infty) = 0$ if models of T do not contain infinite-dimensional cubes (i. e., dimensions of cubes are totally bounded), otherwise we set $\mathrm{Inv}_T(\infty) = 1$.

The *diameter* $d(T)$ of cubic theory T is the maximal distance between elements in models of T, if these distances are bounded, and we set $d(T) \rightleftharpoons \infty$ otherwise. The *support* (accordingly the ∞-*support*) $\mathrm{Supp}(T)$ ($\mathrm{Supp}_\infty(T)$) of theory T is the set $\{n \in \omega \mid \mathrm{Inv}_T(n) \neq 0\}$ ($\{n \in \omega \mid \mathrm{Inv}_T(n) = \infty\}$)

If the diameter $d(T)$ is finite then there exists an upper estimate for dimensions of cubes, being in models of T. It means that $\mathrm{Supp}(T)$ is finite, i. e., $\mathrm{Inv}_T(\infty) = 0$. In this case the ∞-support is non-empty.

If $d(T) = \infty$ then $\mathrm{Inv}_T(\infty) = 1$. In this case the support $\mathrm{Supp}(T)$ can be either finite or infinite.

2. Cubic theories with finite diameters

The following two assertions present the description of spectra for cubic theories having finite diameters.

THEOREM 2.1. *If the diameter $d(T)$ is finite and the ∞-support is a singleton then T is a strongly minimal totally categorical theory.*

PROOF. Note that any model $\mathcal{M}$ of T consists of $|\mathcal{M}|$ connected components and each of them is finite. Moreover, all connected components, except finitely many of them, are pairwise isomorphic. Then, obviously, any two models of cardinality λ are isomorphic ($I(T, \lambda) \equiv 1$) and only finite or cofinite sets in the models are definable by formulas with parameters, i. e., the theory T is strongly minimal. □

THEOREM 2.2. *If the diameter $d(T)$ is finite and the ∞-support contains $k > 1$ elements then $I(T, \omega_0) = 1$, $I(T, \omega_\alpha) = (\alpha + 1)^k - \alpha^k$ for $0 < \alpha < \omega$, and $I(T, \omega_\alpha) = |\alpha|$ for $\alpha \geq \omega$.*

PROOF. Any model $\mathcal{M}$ of T consists again of $|\mathcal{M}|$ connected components and each of them is finite. In this case, for any $n \in \mathrm{Supp}_\infty(T)$ the model $\mathcal{M}$ has infinitely many connected components, being n-cubes, and for any

$n \in \omega \setminus \mathrm{Supp}_\infty(T)$ the number of connected components, being n-cubes, is finite. Hence, any countable model of T contains countably many n-cubes for any $n \in \mathrm{Supp}_\infty(T)$, and the theory T is ω-categorical.

The number of pairwise non-isomorphic models in the cardinality ω_α, where $0 < \alpha < \omega$, is defined by the number of possibilities for distributions of n-cubes for each $n \in \mathrm{Supp}_\infty(T)$. This number is equal to $(\alpha+1)^k - \alpha^k$ since for each k each, of $\omega_0, \ldots, \omega_\alpha$, number of correspondent cubes is possible and at least one of them should give ω_α cubes. Thus $I(T, \omega_\alpha) = (\alpha+1)^k - \alpha^k$ for $0 < \alpha < \omega$. If $\alpha \geq \omega$ then the defining number of possibilities is equal to $|\alpha|$ and $I(T, \omega_\alpha) = |\alpha|$. $\square$

3. Countable models of cubic theories with infinite diameter

Below we consider only cubic theories with the infinite diameter.

As noticed above, the infinite diameter of cubic theory can be accompanied with a finite or infinite support $\mathrm{Supp}(T)$. For these characteristics, the number of connected components, being n-cubes, is constant for any countable model of given cubic theory and the number of connected components having the infinite diameter, by compactness, can have an arbitrary value in $(\omega+1) \setminus \{0\}$. Thus, for the description of spectra and of structures for countable models of cubic theories it suffices to describe structures of countable connected components $\mathcal{C}$ for countable models $\mathcal{M}$ of cubic theories T.

We shall show that all countable connected components are pairwise isomorphic and, whence, are ω-cubes. It suffices to consider countable connected components obtained, by compactness, from finite cubes with unbounded diameters, as well as countable connected components being elementary substructures of infinite connected components for models of cubic theories.

PROPOSITION 3.1. *If T is a theory of model, whose connected components are finite cubes, then any countable connected component of a model of T is an ω-cube.*

PROOF. Let $\mathcal{C}$ be a countable connected component of a model of T. Note that $\mathcal{C}$ cannot have vertices of finite degrees. Indeed, if a vertex a in $\mathcal{C}$ has a finite degree n then this connected component is a (finite) n-cube, since a formula of cubic theory records the information that a vertex of degree n belongs to an n-cube without proper extensions.

Now we consider a vertex a in $\mathcal{C}$ and construct an isomorphism φ between an ω-cube $\mathcal{Q}_f$, for a function $f \in \{0,1\}^\omega$, and the structure $\mathcal{C}$. We put $\varphi_0 \rightleftharpoons \{\langle f, a\rangle\}$ and link bijectively, by a map $\varphi_1 \supset \varphi_0$, the set B of vertices in $\mathcal{Q}_f$, adjacent with f, with the set A of vertices in C adjacent with a. The map φ_1 is an isomorphism between $\mathcal{Q}_f|_{(\{f\}\cup B)}$ and $\mathcal{C}|_{(\{a\}\cup A)}$, since no two vertices in B (respectively in A) are not adjacent.

Since, by 1.9, each vertex in Q_f belongs to the definable closure $\mathrm{dcl}(\{f\}\cup B)$ and any finite set D of vertices in $\mathcal{C}$ is contained in a least finite cube, replacing parameters $b \in \{f\} \cup B$ of formulas, defining $\mathrm{dcl}(\{f\} \cup B)$ and having unique solutions, by parameters $\varphi_1(b)$ we obtain formulas defining an ω-cube being a substructure of $\mathcal{C}$ with the set $\mathrm{dcl}(\{a\} \cup A)$ of vertices. The transformation for unique solutions of formulas $\theta(x, \bar{b})$ to unique solutions of formulas $\theta(x, \varphi_1(\bar{b}))$ defines an isomorphism $\varphi_2 \supset \varphi_1$ between $\mathcal{Q}_f$ and $\mathcal{C} \upharpoonright \mathrm{dcl}(\{a\} \cup A)$. It remains to note that $C = \mathrm{dcl}(\{a\} \cup A)$. Indeed, each vertex d in $\mathcal{C}$ is contained in the least cube $\mathcal{C}_0 \subset \mathcal{C}$ including a and d, and any vertex in $\mathcal{C}_0$, in particular, d, belongs to the definable closure of $\{a\} \cup (C_0 \cap A)$. Thus, φ_2 is the required isomorphism φ. □

Proposition 3.2. *If $\mathcal{C}$ is an infinite connected component for a model of cubic theory T then each countable elementary substructure of $\mathcal{C}$ is an ω-cube.*

Proof essentially repeats the proof of Proposition 3.1. Considering an ω-cube $\mathcal{Q}_f$ for a function $f \in \{0,1\}^\omega$ and choosing an arbitrary vertex a in a countable elementary substructure $\mathcal{C}_0$ of $\mathcal{C}$ we construct an isomorphism between $\mathcal{Q}_f$ and $\mathcal{C}_0$ mapping f to a, the set B of vertices in $\mathcal{Q}_f$, adjacent to f, onto the set A of vertices in $\mathcal{C}_0$ adjacent with a, and the vertices in $\mathrm{dcl}(\{f\} \cup B)$ to the correspondent vertices in $\mathrm{dcl}(\{a\} \cup A)$. □

Recall that a *disjunctive union* $\bigsqcup\limits_{i\in I} \mathcal{M}_i$ of pairwise disjoint structures $\mathcal{M}_i$, $i \in I$, of predicate language L is the structure of the language L with the universe $\bigsqcup\limits_{i\in I} M_i$ and the interpretations of predicate symbols in L as the unions of their interpretations in $\mathcal{M}_i$, $i \in I$.

Propositions 3.1 and 3.2 imply

Corollary 3.3. *If $\mathcal{M}$ is a countable model of cubic theory T having the infinite diameter, $\mathcal{C}$ is a countable connected component of a model of a cubic theory T', disjoint with $\mathcal{M}$, then $\mathcal{C}$ is an ω-cube and $\mathcal{M} \prec \mathcal{M} \bigsqcup \mathcal{C}$.*

By Corollary 3.3 we have the following

Theorem 3.4. *If T is a cubic theory with the infinite diameter then any countable model of T is represented by at most countable disjoint union of ω-cubes united by at most countable disjoint union of n-cubes for $n \in \omega$. The number of n-cubes in the disjoint union does not depend on choice of countable model of T and the number of ω-cubes varies from 0 to ω inclusive, if $|\mathrm{Supp}(T)| = \omega$, and from 1 to ω inclusive, if $|\mathrm{Supp}(T)| < \omega$. The isomorphism type of a countable model $\mathcal{M}$ of T is defined by the number of ω-cubes being connected components of $\mathcal{M}$.*

Theorem 3.4 immediately implies

Corollary 3.5. *If the diameter $d(T)$ of cubic theory T is infinite then $I(T, \omega) = \omega$.*

Recall [10] that a model $\mathcal{M}$ of a theory T is called *limit* if $\mathcal{M}$ is not prime over tuples and it is represented as a union of countable elementary chain of models of T, being prime over tuples. A theory T is *l-categorical* if T has a unique, up to isomorphism, limit model.

Since countable models of a cubic theory are prime over tuples if and only if they contain finitely many ω-cubes and all countable models of a cubic theory and with countably many countable connected components are pairwise isomorphic, we have

Proposition 3.6. *Any cubic theory with the infinite diameter is l-categorical.*

Below we define a set of formulas of cubic theory T basing all types in $S(T)$. For this aim, we use the syntactic approach for the construction of generic structures [8] and a schema for the construction of generic structures caused by this approach and presented in [9].

Since each finite connected component for a model of cubic theory has the cardinality 2^n for some $n \in \omega$, and, having the same finite cardinality, connected components are isomorphic to cubes of correspondent dimension, the description of types for finite sets A contained in finite cubes is defined by principal formulas describing that elements of A (do not) belong to common n-cubes (i. e., by formulas $\rho_k(x,y)$ describing lengthes k of shortest routes between elements in A and by formulas $\rho_\infty(x,y)$ "saying" that elements in A do not belong to common connected components) as well as the relationship of elements in A, belonging to a common n-cube $\mathcal{C}$, in this n-cube $\mathcal{C}$ (defined by *projections* $\exists\bar{y}\chi(\bar{x},\bar{y})$ of formulas $\chi(\bar{x},\bar{y})$, where $\chi(\bar{a},\bar{b})$ is a conjunction of formulas forming the diagram of n-cube $\mathcal{C}$, $\bar{a}$ is a tuple of all elements in $\mathcal{C}$ belonging to A, $\bar{b}$ is a tuple of elements in $\mathcal{C}$ that do not belong to A). We shall consider the basing of types up to these descriptions and therefore below, up to and including Theorem 3.8, we shall assume that T does not have models with finite connected components. By Theorem 3.4, such a theory is unique and it is the theory of ω-cube.

Now, let T be the theory of ω-cube, $\mathcal{M}$ be a countable model of T, and A be a finite set in $\mathcal{M}$. The structure $\mathcal{A}$ of graph language $\{R\}$, consisting of the universe A, the relation R on A, being the restriction of the graph relation on $\mathcal{M}$ to the set A, and the record W, is the *c-graph* if W consists of the following formulas:

(1) $\rho_k(a_i,a_j)$ describing lengthes k of shortest routes between elements a_i and a_j in A;

(2) $\neg\rho_k(a_l,a_m)$, $k\in\omega$, "saying" that elements a_l and a_m of A belong to distinct connected components;

(3) projections $\exists\bar{y}\chi(\bar{a},\bar{y})$ of conjunctions $\chi(\bar{a},\bar{b})$ of formulas forming diagrams of n-cubes $\mathrm{FC}(\Gamma_0)$, where Γ_0 are maximal, by inclusion, subgraphs of ω-cubes in $\mathcal{M}$ consisting of elements in A.

The c-graph $\mathcal{A}$ is denoted by $\langle A,R,W\rangle$. We suppose that the empty structure $\langle\varnothing,\varnothing,\varnothing\rangle$ is also a c-graph.

For the c-graph $\mathcal{A}=\langle A,R,W\rangle$ we denote by $\mathrm{cc}(\mathcal{A})$ a minimal, by inclusion, graph $\Gamma\supseteq\langle A;R\rangle$ containing all shortest (a,b)-routes for each pair $(a,b)\in A^2$ connected by a route in accordance with the record W.

Using assertions of Section 1 it is easy to see that $\mathrm{cc}(\mathcal{A})$ is represented as a disjunctive union of finite cubes of form $\mathrm{FC}(\Gamma_0)$ lying in distinct maximal ω-cubes of $\mathcal{M}$. Whence, any graph $\mathrm{cc}(\mathcal{A})$ consists of connected components, each of which is a n-cube. The elements of $\mathrm{cc}(\mathcal{A})$ form the algebraic closure $\mathrm{acl}(A)$, and $\mathrm{acl}(A)$ is represented as the union of algebraic closures $\mathrm{acl}(C\cap A)$ for all connected components $\mathcal{C}$ of $\mathcal{M}$ having elements in A. The cubes forming $\mathrm{acl}(A)$ are said to be the *cubes generated by* A. Note also that for each c-graph $\mathcal{A}$ lying in a connected component, $|\mathrm{acl}(A)|=2^d$, where d is the diameter of $\mathrm{cc}(\mathcal{A})$.

We define the relation $\subseteq_c$ on the class of c-graphs. A c-graph $\mathcal{A} = \langle A, R_{\mathcal{A}}, W_{\mathcal{A}}\rangle$ is called a *c-subgraph* of a c-graph $\mathcal{B} = \langle B, R_{\mathcal{B}}, W_{\mathcal{B}}\rangle$ (written $\mathcal{A} \subseteq_c \mathcal{B}$) if $A \subseteq B$, $R_{\mathcal{A}} = R_{\mathcal{B}} \cap A^2$, and $W_{\mathcal{A}}$ is the record correspondent the conditions 1–3 on shortest routes and on projections of diagrams linking elements of A in a graph $\mathrm{cc}(\mathcal{B})$.

A c-graph $\mathcal{A} = \langle A, R, W\rangle$ is called *closed* if $\mathcal{A} = \mathrm{cc}(\mathcal{A})$, i. e., $\mathcal{A}$ contains all shortest routes and diagrams written in W. We denote by $\mathbf{K}_0$ the class of all closed c-graphs.

If $\mathcal{A}$, $\mathcal{B} = \langle B, R_{\mathcal{B}}, W_{\mathcal{B}}\rangle$, and $\mathcal{C} = \langle C, R_{\mathcal{C}}, W_{\mathcal{C}}\rangle$ are closed c-graphs, $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$, then a *free c-amalgam* of c-graphs $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$ (denoted by $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$) is a minimal, by inclusion, closed c-graph containing the structure $\langle B \cup C, R_{\mathcal{B}} \cup R_{\mathcal{C}}, W_{\mathcal{B}} \cup W_{\mathcal{C}}\rangle$.

If $\mathcal{A}$ is a $n_{\mathcal{A}}$-cube, $\mathcal{B}$ is a $n_{\mathcal{B}}$-cube, and $\mathcal{C}$ is a $n_{\mathcal{C}}$-cube then the $(n_{\mathcal{B}} + n_{\mathcal{C}} - n_{\mathcal{A}})$-cube $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$ can be constructed as follows. Assuming that the vertices of $\mathcal{A}$ are represented by sequences $(0, \ldots, 0, \delta_1, \ldots, \delta_{n_{\mathcal{A}}}, 0, \ldots, 0)$ (where the first zeros are repeated $(n_{\mathcal{B}} - n_{\mathcal{A}})$ times and the second zeros $(n_{\mathcal{C}} - n_{\mathcal{A}})$ times), the vertices of $\mathcal{B}$ by $(\delta_1, \ldots, \delta_{n_{\mathcal{B}}}, 0, \ldots, 0)$ (where zeros are repeated $(n_{\mathcal{C}} - n_{\mathcal{A}})$ times), and vertices of $\mathcal{C}$ by $(0, \ldots, 0, \delta_1, \ldots, \delta_{n_{\mathcal{C}}})$ (where zeros are repeated $(n_{\mathcal{B}} - n_{\mathcal{A}})$ times), $\delta_i \in \{0, 1\}$, we can take for $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$ the cube with the universe $\{0, 1\}^{(n_{\mathcal{B}} + n_{\mathcal{C}} - n_{\mathcal{A}})}$ and the identical embedding of $\mathcal{B}$ and $\mathcal{C}$ to this cube.

Thus, any free c-amalgam $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$ is formed by n-cubes, generated by cubes included in $\mathcal{B}$ and $\mathcal{C}$.

An injective map $f\colon A \to B$ is called a *c-embedding* of c-graph $\mathcal{A} = \langle A, R_{\mathcal{A}}, W_{\mathcal{A}}\rangle$ to a c-graph $\mathcal{B} = \langle B, R_{\mathcal{B}}, W_{\mathcal{B}}\rangle$ (written $f\colon \mathcal{A} \to_c \mathcal{B}$) if f is an embedding of the graph $\langle A, R_{\mathcal{A}}\rangle$ to the graph $\langle B, R_{\mathcal{B}}\rangle$ such that the record $W_{f(A)}$ of the c-subgraph of $\mathcal{B}$ with the universe $f(A)$ is equal to the record obtained from W_A with the replacement of all elements $a \in A$ by the elements $f(a)$.

c-Graphs $\mathcal{A}$ and $\mathcal{B}$ are called *c-isomorphic* if there exists a c-embedding $f\colon \mathcal{A} \to_c \mathcal{B}$ with $f(A) = B$. The map f is called a *c-isomorphism* between $\mathcal{A}$ and $\mathcal{B}$ and the c-graphs $\mathcal{A}$ and $\mathcal{B}$ are *c-isomorphic copies.*

LEMMA 3.7 (Amalgamation Lemma). *The class $\mathbf{K}_0$ of all closed c-graphs satisfies the c-amalgamation property* (c-AP)*, i. e., for any c-embeddings $f_0\colon \mathcal{A} \to_c \mathcal{B}$ and $g_0\colon \mathcal{A} \to_c \mathcal{C}$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C} \in \mathbf{K}_0$, there is a c-graph $\mathcal{D} \in \mathbf{K}_0$ and c-embeddings $f_1\colon \mathcal{B} \to_c \mathcal{D}$ and $g_1\colon \mathcal{C} \to_c \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.*

Proof. Without loss of generality we may assume that $\mathcal{A} \subseteq_c \mathcal{B}$ and $\mathcal{A} \subseteq_c \mathcal{C}$. Clearly, the closed c-graph $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$ can be taken for $\mathcal{D}$. □

Denote by $\mathbf{T}_0$ the class of all types correspondent to c-graphs in $\mathbf{K}_0$. Using Lemma 3.7 one can immediately check that the class $\mathbf{T}_0$ with the relation $\leqslant$, where $\Phi(A) \leqslant \Psi(B) \Leftrightarrow \mathcal{A} \subseteq_c \mathcal{B}$, is a self-sufficient generic class having the uniform t-amalgamation property [8].

Hence, there is a $(\mathbf{T}_0; \leqslant)$-generic structure $\mathcal{N}$, its theory coincides with the theory T, and Theorem 4.1 in [8] implies

Theorem 3.8. *The $(\mathbf{T}_0; \leqslant')$-generic structure $\mathcal{N}$ is saturated. Any finite set $A \subseteq N$ is extensible to its self-sufficient closure $\overline{A} = \mathrm{cc}(A) \subseteq N$, and the type $\mathrm{tp}(\overline{A})$ is implied by its subtype consisting of quantifier free formulas describing cubes $\mathcal{C}$ composed by elements of $\overline{A}$, as well as of formulas with two free variables describing that any two elements in distinct cubes $\mathcal{C}$ are not linked by routes.*

The generic structure $\mathcal{N}$ consists of countably many pairwise isomorphic connected components, each of which is an ω-cube.

Since every ω-cube has a proper extension, being again an ω-cube, some countable extension of a countable model $\mathcal{M}$ for a cubic theory T has realizations of all types in $S^1(M)$. It means that any cubic theory is ω-stable.

Moreover, by Theorem 3.8 the equality $\mathrm{MR}(\varphi_n(x, a)) = n$ holds , where $\varphi_n(x, y)$ is a formula "saying" that the distance between x and y is equal to n. Since each connected component $\mathcal{C}$ of a model of cubic theory is represented as the union $\bigcup\limits_{n\in\omega} \varphi_n(\mathcal{C}, a)$, $a \in C$, we have $\mathrm{RM}(x \approx x) = \omega$.

Thus we obtain

Theorem 3.9. *Any cubic theory with the infinite diameter is ω-stable of Morley rank ω.*

Recall [3] that a countable theory T of a predicate language is called 1-*locally countably categorical*, or a LCC1-*theory*, if T has finitely many non-principal 1-types $p_1(x), \ldots, p_n(x)$, and for any formulas $\varphi_i(x) \in p_i(x)$, $i = 1, \ldots, n$, and for any tuple $\bar{a}$, for which every coordinate realize some type p_i, the structure, defined by formulas with parameters in $\bar{a}$ on a set defined by formula $\neg\varphi_1(x) \wedge \ldots \wedge \neg\varphi_n(x)$, is ω-categorical.

Note that any cubic theory T has a non-principal 1-type in $S(T)$ if and only if the support $\mathrm{Supp}(T)$ is infinite. In this case, the unique non-principal 1-type $p(x)$ is the type of elements with the infinite eccentricity. Moreover, each formula, negating the excess of given natural number by eccentricity, defines a countably categorical structure on a set of its solutions. Thereby we have

THEOREM 3.10. *Any cubic theory is* 1*-locally countably categorical.*

4. Spectra for uncountable models of cubic theories with infinite diameter

Repeating the proof of Propositions 3.1 and 3.2 with the replacement of countable connected component $\mathcal{C}$ by component of cardinality $\lambda > \omega$ we obtain

PROPOSITION 4.1. *If T is the theory of model, whose connected components are finite cubes, then any infinite connected component of a model of T, having a cardinality λ, is a λ-cube.*

PROPOSITION 4.2. *If $\mathcal{C}$ is an infinite connected component of a model of cubic theory T then each its elementary substructure of cardinality λ is a λ-cube.*

Propositions 4.1 and 4.2 imply

COROLLARY 4.3. *If $\mathcal{M}$ is a models of cubic theory T having the infinite diameter, $\mathcal{C}$ is an infinite connected component of a model of cubic theory T', having the cardinality λ and disjoint with $\mathcal{M}$, then $\mathcal{C}$ is a λ-cube and $\mathcal{M} \prec \mathcal{M} \bigsqcup \mathcal{C}$.*

By Corollary 4.3, we obtain

THEOREM 4.4. *If T is a cubic theory having the infinite diameter then any model $\mathcal{M}$ of T and of cardinality $\lambda > \omega$ is represented as a disjoint union of at most λ pairwise isomorphic μ-cubes for each $\omega \le \mu \le \lambda$, being disjointly united with at most λ of n-cubes for $n \in \omega$. For each natural number n a finite number of n-cubes used in the disjoint union does not depend on choice of $\mathcal{M}$ and infinite numbers of n-cubes may vary, independently of each other, from ω to λ. The numbers of μ-cubes, $\omega \le \mu \le \lambda$, vary, independently of each other, from* 0 *to λ.*

By Theorem 4.4, the number $I(T,\lambda)$ for a cubic theory T with the infinite diameter is defined by the numbers of connected components having distinct finite cardinalities, as in Theorems 2.1 and 2.2, as well as by the numbers of connected components having infinite cardinalities $\mu\leq\lambda$.

THEOREM 4.5. *If the diameter $d(T)$ of cubic theory T is infinite and α is a nonzero ordinal then $I(T,\omega_\alpha)=\max\left\{2^{|{\rm Supp}_\infty(T)|},|\omega+\alpha|^{|\alpha|}\right\}$.*

PROOF. The number of pairwise non-isomorphic models of cardinality ω_α, where $\alpha>0$, is defined by variants for distributions of numbers of n-cubes for each $n\in{\rm Supp}_\infty(T)$ as well as by variants for distributions of numbers of ω_β-cubes for each $\beta\leq\alpha$.

Since the number of n-cubes for each $n\in{\rm Supp}_\infty(T)$ varies from ω_0 to ω_α, the number of finite cubes gives:

- finitely many variants for $|{\rm Supp}_\infty(T)|<\omega$ and $\alpha<\omega$;
- $|\alpha|$ variants for $|{\rm Supp}_\infty(T)|<\omega$ and $\alpha\geq\omega$;
- 2^ω variants for $|{\rm Supp}_\infty(T)|=\omega$ and $\alpha<\omega$;
- $2^{|\alpha|}$ variants for $|{\rm Supp}_\infty(T)|=\omega$ and $\alpha\geq\omega$.

The number of ω_β-cubes, $\beta\leq\alpha$, may vary from 0 to ω_α and we get $|\omega+\alpha|^{|\alpha|}$ variants by cardinalities from ω_0 to ω_α for connected components. Having the number of variants for finite and infinite cubes we conclude that $I(T,\omega_\alpha)=\max\left\{2^{|{\rm Supp}_\infty(T)|},|\omega+\alpha|^{|\alpha|}\right\}$, $\alpha>0$. □

5. On spectra of models for cubic theories with colored edges

A *coordinated coloring for edges* of a canonical representative $\mathcal{Q}_f$ of a μ-cube is a setting of colors for edges in $\mathcal{Q}_f$ such that edges $[f_1,f_2]$ and $[f_1',f_2']$ in $\{0,1\}^\mu$ have the same color if there is a coordinate i, for which f_1 and f_2 differ by i as well as f_1' and f_2' differ by i.

Any coordinated coloring provides a partition of the set of all positions for coordinates of vertices by equivalence classes I_j as well as an expansion by binary predicates R_{I_j} refining the relation R with the following rule:

$$(f_1,f_2)\in R_{I_j}\Leftrightarrow(f_1,f_2)\in R\text{ and }f_1(i)\neq f_2(i)\text{ for some }i\in I_j.$$

In this case, the cube $\mathcal{Q}_f$ is called *coordinated edge-colored*.

A *coordinated coloring for edges* of a μ-cube $\mathcal{Q}$ is a setting of colors for edges, i. e., an expansion of $\mathcal{Q}$ by predicates R_{I_j} such that the expanded μ-cube is isomorphic to a coordinated edge-colored canonical representative $\mathcal{Q}_f$.

Coordinated colorings for edges of μ_1-cubes $\mathcal{Q}_f$ and of μ_2-cube $\mathcal{Q}_g$ (where $\mu_1 \leq \mu_2 \leq \omega$) are *agreed* if the partition for coordinates of vertices to the equivalence classes I_j in $\mathcal{Q}_g$, restricted to μ_1, coincides with the partition for coordinates of vertices to the equivalence classes I_j in $\mathcal{Q}_f$. In this case, the coordinated colors for edges of μ_2-cube $\mathcal{Q}_g$ and μ_1-cube $\mathcal{Q}_f$ are also *agreed*.

Coordinated colorings for edges of connected components $\mathcal{C}_1$ and $\mathcal{C}_2$ of a model $\mathcal{M}$ of a cubic theory are *agreed* if the coordinated colors for edges of canonical representatives of these connected components are agreed.

An *agreed coordinated coloring for edges* of model $\mathcal{M}$ is a family of pairwise agreed coordinated coloring for edges of all connected components of $\mathcal{M}$. In this case, the model $\mathcal{M}$ is *agreed coordinated edge-colored* of a ACEC-*model*, and its theory is *agreed coordinated edge-colored cubic theory*, or a ACEC-*theory*.

Note that having an agreed coordinated coloring we obtain a monotonic non-decreased functions $g_{R_{I_j}}$ signifying the number of edges of colors R_{I_j} incident to chosen vertices according to dimensions n of n-cubes containing these vertices.

Below we consider countable ACEC-models $\mathcal{M}$ of cubic theories with the infinite diameter, of countable language, and having infinitely many n-cubes for each $n \in \omega$ such that every function $g_{R_{I_j}}$ increases unboundedly for $n \to \infty$.

Note that, in any ACEC-model, any two monochromatic edges lying in cubes of the same dimension are linked by an automorphism, and any μ_1-cube is embeddable in any μ_2-cube for $\mu_1 \leq \mu_2$. Since any vertex of given cube has an incident edge of every color presented in this cube, 1-types over $\emptyset$ are defined by eccentricities of realizing vertices. Thus all n-cubes, for fixed n, have the same type.

Almost word for word repeating the proof of Theorems 3.9 and 3.10 for ACEC-theories of specified models, we obtain the following theorem.

THEOREM 5.1. *There is a theory T of countable binary language $\{R_{I_j} \mid j \in \omega\}$, satisfying the following conditions:*

(1) *T is an agreed coordinated edge-colored cubic theory of the infinite diameter and with models having infinitely many n-cubes for each $n \in \omega$ such that each function $g_{R_{I_j}}$, signifying the number of edges of color R_{I_j} incident to chosen vertices according to dimensions n of n-cubes, increases unboundedly for $n \to \infty$;*

(2) *T is a 1-locally countably categorical ω-stable theory of Morley rank ω, having countably many pairwise non-isomorphic countable models.*

Thus, in particular, there is an ω-stable LCC1-theory with colored edges such that the type of each element is defined by dimension of maximal cube containing this element or, equivalently, by eccentricity of the element, and each vertex realizing the unique non-principal 1-type has infinitely many incident edges for each of countably many colors.

Models of ACEC-theories admit the spectral description similar the description in Sections 3, 4 and dependent on functions $g_{R_{I_j}}$.

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