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CONTRACTION CONTRACTED

Abstract

This short article is mainly of methodological character. We are concerned with the problem of elimination of the rule of contraction from the set of primitive rules of a sequent calculus. It is desirable since this rule is technically embarassing in cut elimination proofs. It appears that simple change in the way of reading composition of contexts in sequents allows for eliminating contraction in some types of sequent calculi. In particular we do not need to prove it as an admissible rule in contrast to what is deserved in known Dragalin's construction. For simplicity sake we consider only the case of propositional classical logic but the proposed solution may be applied also to sequent formalizations of other logics, in particular to extensions of classical logic.

1. Introduction

It is well known that in sequent calculi the contexts Γ, Δ in any sequent $\Gamma \Rightarrow \Delta$ may be read in different ways. The most popular proposals[1] are:

- sets (of formulae)
- multisets
- sequences

The original reading due to Gentzen is the third one. It is also the most demanding on the side of rules since in order to obtain a formalization

[1]Sometimes more elaborate readings are necessary, e.g. for excluding associativity which is implicit in the three standard approaches.

of classical logic we need structural rules of permutation (exchange) and contraction.

$$(P\Rightarrow)\ \frac{\Pi, \varphi, \psi, \Gamma \Rightarrow \Delta}{\Pi, \psi, \varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow P)\ \frac{\Gamma \Rightarrow \Delta, \psi, \varphi, \Pi}{\Gamma \Rightarrow \Delta, \varphi, \psi, \Pi}$$

$$(C\Rightarrow)\ \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow C)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

In fact, using sequences is rarely necessary[2]. From the remaining two choices it seems that using sets makes things easier, especially in semantical analyses (in particular, both permutation and contraction rules simply dissapear). However, in the proof-theoretic considerations multisets are preferable (see Negri and von Plato [2011] pp. $89-91$ for arguments against set-approach). Recall that multisets are sets where the number of occurrences of the same element counts, e.g. $\{a, a, b\}$ and $\{a, b\}$ are different multisets. Formally multisets may be defined as pairs $\langle X, f\rangle$, where X is an ordinary set and $f : X \longrightarrow \mathbb{N}$. In this approach permutation rules are superfluous but contraction rules are treated as indispensable.

Why do we need contraction? It is sometimes necessary to decrease the number of occurrences of a formula in a sequent. There are at least 4 situations where uncontrolled multiplication of occurrences of the same formula may take place:

1. Multiple occurrences of the same formula may be present already in axiomatic sequents if they have generalised form, i.e. instead of the simple form $\varphi \Rightarrow \varphi$ they have additional formulae (context) in the antecedent or consequent.

2. Multiplication may be introduced by the application of Weakening rules, of the form:

$$(W\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow W)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

3. Multiplication may be introduced via application of some logical rule which introduces into antecedent (or consequent) a compound formula which is already present in the context on this side of the sequent.

4. The application of (some kind of, i.e. context-free) two-premise rules, including Cut, which leads to composition of the contexts of premise-sequents.

[2]The most important example requiring sequences is Lambek Calculus (see Paoli [2002]).

Because of that we may need contraction to diminish the number of occurrences of the same formula. The first two mechanisms of generating multiple occurrences are easily controlled, and moreover they appear only in systems where either axioms are generalised (with contexts) or weakening is primitive. In this paper we focus on the fourth source of multiplication of formulae.

But why are we interested in elimination of contraction? It is well known that the rule of contraction may cause troubles, especially in the context of proofs of cut eliminability and decidability[3]. Many solutions of different kind were offered for this problem. Gentzen [1934] to avoid the complications connected with contraction introduced a special rule Mix (or Multicut) instead of Cut. Curry [1963] provided a proof of cut elimination where global transformations of proofs are defined. Dragalin [1988] depending on Ketonen's rules provided a system with no structural rules at all and allowing a proof of elimination directly for Cut, but in order to obtain the result he had to show first the admissibility of contraction, i.e. that the application of these rules always leads to sequents provable by means of primitive rules of the calculus. Recently Negri and von Plato [2013] provided a system where deletion of new introduced copies of formulas is implicitly introduced into rules formation.

Let us focus on Dragalin's solution. In his system contraction (and also weakening) are not primitive but the proof of cut admissibility presupposes three actions:

1. A calculus requires special form of axiomatic sequents, i.e. atomic with contexts: $\Gamma, \varphi \Rightarrow \varphi, \Delta$, where φ stands for atomic formula. Clearly we must also show that this restriction does not preclude a proof of such sequents for any compound φ.

2. Next we must show height-preserving admissibility of weakening rules. Height-preserving admissibility means that the proof obtained by means of admissible rule is not longer, i.e. if $\Gamma \Rightarrow \Delta$ has a proof of height n, then the proof of, say, $\Gamma \Rightarrow \Delta, \varphi$ is of height $\leq n$. The height of a proof of a sequent is counted as the number of sequents in the longest branch of this proof.

3. We must show also the height-preserving invertibility of logical rules. It means that for each logical rule if an instance of the conclusion is provable with a proof of height n, then instances of premises of this rule are also

[3] In Paoli [2002] one can find other reservations of philosophical nature.

provable with height $\leq n$.

It seems that even if the proof of cut admissibility is simpler, then the necessary preliminaries are many and complicated (see the exposition in Troelstra and Schwichtenberg [1996], or Negri and von Plato [2001]). Moreover, contraction is not eliminable in the strong sense of not being addressed at all, but only in the weaker sense of being admissible not primitive. The proof of admissibility of Cut still requires admissibility of contraction, as we will see in the next section. It seems however, that things can be simplified if we change the way of composing contexts in conclusions of two-premise rules.

2. Where contraction counts?

Let us recall that logical rules may appear in two forms:

- multiplicative (context-free, intensional, internal, group-theiretical)
- additive (context-sharing, extensional, combining, lattice-theoretical)

For example multiplicative rules (M-rules) for $\wedge$ looks like that:

$$(\Rightarrow\wedge)\ \frac{\Gamma\Rightarrow \Delta, \varphi \qquad \Pi\Rightarrow \Sigma, \psi}{\Gamma, \Pi\Rightarrow \Delta, \Sigma, \varphi\wedge\psi} \qquad\qquad (\wedge\Rightarrow)\ \frac{\varphi, \psi, \Gamma\Rightarrow \Delta}{\varphi\wedge\psi, \Gamma\Rightarrow \Delta}$$

Additive rules (A-rules) are the following:

$$(\Rightarrow\wedge)\ \frac{\Gamma\Rightarrow \Delta, \varphi \qquad \Gamma\Rightarrow \Delta, \psi}{\Gamma\Rightarrow \Delta, \varphi\wedge\psi} \qquad\qquad (\wedge\Rightarrow)\ \frac{\varphi, \Gamma\Rightarrow \Delta}{\varphi\wedge\psi, \Gamma\Rightarrow \Delta} \qquad\qquad \frac{\psi, \Gamma\Rightarrow \Delta}{\varphi\wedge\psi, \Gamma\Rightarrow \Delta}$$

In fact these rules define different connectives in the absence of some structural rules which gave an impetus to develop proof theory of substructural logics. However, in case of classical logic (to which we restrict our considerations) both pairs of rules are interderivable. To derive A-rules from M-rules we need contraction for two-premise rules and weakening for one-premise rules; in the opposite direction we need weakening (for two-premise rules) and contraction (for one-premise rules). Hence one can obtain 4 types of sequent calculi for classical logic by taking either only M-rules or A-rules or by taking different set of rules for two-premise rules and different for one-premise rules (Of course some mixed combination are also possible by taking different sets for different constants - e.g. Gentzen's LK where $\vee$ and $\wedge$ are characterised by A-rules, whereas $\rightarrow$ is characterised by

M-rules). Similarly we can consider Cut in either multiplicative or additive form i.e.:

$$(M-Cut)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (A-Cut)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

By addition of Cut in either of two forms we obtain 8 basic types of calculi.

At first it seems that whichever type of sequent calulus we consider contraction rules are unavoidable (either as primitive or admissible) since:

1. some theses are not cut-free derivable without contraction if one-premise rules are additive (e.g. LEM, Noncontradiction Law);
2. some theses are not cut-free derivable without contraction if two-premise rules are multiplicative (e.g. Peirce Law, Absorption Law).

Let us illustrate these two points:

$$(\Rightarrow C)\ \dfrac{(\Rightarrow \vee)\ \dfrac{(\Rightarrow \vee)\ \dfrac{(\Rightarrow \neg)\ \dfrac{p \Rightarrow p}{\Rightarrow p, \neg p}}{\Rightarrow p \vee \neg p, \neg p}}{\Rightarrow p \vee \neg p, p \vee \neg p}}{\Rightarrow p \vee \neg p}$$

$$(\Rightarrow\to)\ \dfrac{(\Rightarrow C)\ \dfrac{(\to\Rightarrow)\ \dfrac{(\Rightarrow\to)\ \dfrac{p \Rightarrow p, q}{\Rightarrow p, p \to q} \qquad p \Rightarrow p}{(p \to q) \to p \Rightarrow p, p}}{(p \to q) \to p \Rightarrow p}}{\Rightarrow ((p \to q) \to p) \to p}$$

In the first example additive $(\Rightarrow \vee)$ was applied; in the second multiplicative $(\to\Rightarrow)$. We can find contraction-free proofs of these theses but with the application of A-cut. The reader may easily check (by exhausting all possibilities of backward proof search) that without cut we cannot provide contraction-free proofs for them.

It seems that because of the above if we want to avoid contraction as primitive we are left with one choice: a system which consists of additive two-premise rules and multiplicative one-premise rules for which either M-cut or A-cut is shown to be admissible. In fact, Ketonen's version of sequent calculus, for which Dragalin proved (M-)cut admissibility is of this kind. In such a system we do not need contraction as primitive but anyway we need it later in the proof of M-cut admissibility. Here is an illustration with reduction of cut-formula complexity:

$$(Cut)\ \dfrac{(\Rightarrow\wedge)\ \dfrac{\Gamma\Rightarrow\Delta,\varphi \qquad \Gamma\Rightarrow\Delta,\psi}{\Gamma\Rightarrow\Delta,\varphi\wedge\psi} \qquad \dfrac{\varphi,\psi,\Pi\Rightarrow\Sigma}{\varphi\wedge\psi,\Pi\Rightarrow\Sigma}\ (\wedge\Rightarrow)}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}$$

this application of Cut is transformed into:

$$\dfrac{\dfrac{\Gamma\Rightarrow\Delta,\psi \qquad \dfrac{\Gamma\Rightarrow\Delta,\varphi \qquad \varphi,\psi,\Pi\Rightarrow\Sigma}{\psi,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\ (Cut)}{\Gamma,\Gamma,\Pi\Rightarrow\Delta,\Delta,\Sigma}\ (Cut)}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}\ (C)$$

where contraction is necessary in the last step.

Moreover – as we noted in the preceding section – in order to provide such a calculus with contraction admissible we must first perform other actions – namely: restrict axioms to atomic with contexts which is necessary to prove height-preserving admissibility of weakening and height-preserving invertibility of logical rules. On the basis of these results we can obtain height-preserving admissibility of contraction and finally apply it in the proof of cut admissibility.

3. Multiset Union

Is it possible to avoid these preliminary complications and to keep the proof of cut admissibility as simple as in Dragalin's approach? We want to sketch some solution based on a different reading of combining multisets. Commonly in the multiset setting[4] combinations of contexts from two premises, i.e. Γ,Δ is treated with no reflexion as a concatenation of two multisets which gives the same effect (modulo innesential order) as additive union, or sum, of (two) multisets which may be characterised as follows:

$\varphi\in^n\Gamma\uplus\Delta$ iff ($\varphi\in^n\Gamma$ and $\varphi\notin\Delta$) or ($\varphi\in^n\Delta$ and $\varphi\notin\Gamma$) or ($\varphi\in^k\Gamma$ and $\varphi\in^l\Delta$ and $n=k+l$)

where $\varphi\in^k\Gamma$ means that k copies of φ belongs to multiset Γ.

[4] For more on multisets see Blizard [1989].

In fact, reading the combination of contexts in terms of concatenation is the only reasonable reading for sequences but for multisets we can apply different solution and use ordinary multiset union which may be characterised as follows:

$\varphi \in^n \Gamma \sqcup \Delta$ iff ($\varphi \in^n \Gamma$ and $\varphi \notin \Delta$) or ($\varphi \in^n \Delta$ and $\varphi \notin \Gamma$) or ($\varphi \in^k \Gamma$ and $\varphi \in^l \Delta$ and $n = max(k, l)$)

The second choice seems to have some advantages which we will try to show. The essential idea is based on the fact that $\Gamma \sqcup \Gamma = \Gamma$, i.e. $\sqcup$ is idempotent in contrast to $\uplus$. In what follows we will write $\Gamma\Delta$ for multiset union hence $\Gamma\Delta, \varphi$ is read as the multiset union of Γ and Δ with displayed occurrence of φ not being an element of $\Gamma\Delta$ (to put it formally $\Gamma\Delta, \varphi$ is a shortcut for $(\Gamma \sqcup \Delta) \uplus [\varphi]$, where $[\varphi]$ denotes the multiset with only one element φ).

4. Advantages

What do we gain by introducing multiset union to two-premise rules (including cut) instead of additive union?

Let us define a sequent calculus **CPLM** for classical propositional logic based on this reading of multiset combination as the one having:

1. simple atomic axioms of the form $\varphi \Rightarrow \varphi$, with φ any propositional variable;
2. both weakening rules;
3. multiplicative cut;
4. all logical rules in multiplicative version.

Note that for all two-premise rules it means that the composition of contexts from premises in the conclusion is a multiset union. We will show it both graphically and by addition of $*$ to the names of rules for distinguishing them from ordinary two-premise M-rules. Hence, for example, we have:

$$(M - Cut*)\ \frac{\Gamma\Rightarrow \Delta, \varphi \qquad \varphi, \Pi\Rightarrow \Sigma}{\Gamma\Pi\Rightarrow \Delta\Sigma} \qquad (\Rightarrow \wedge *)\ \frac{\Gamma\Rightarrow \Delta, \varphi \qquad \Pi\Rightarrow \Sigma, \psi}{\Gamma\Pi\Rightarrow \Delta\Sigma, \varphi \wedge \psi}$$

and similarly for $(\vee \Rightarrow *)$ and $(\rightarrow\Rightarrow *)$.

One can esily check that such a calculus provides adequate formalization of classical propositional logic. All the rules are sound in the same way as their ordinary M-counterparts are. In order to prove completeness one must first prove:

LEMMA 1. *$\varphi \Rightarrow \varphi$ is cut-free provable for any formula φ*

The proof is standard by induction on the complexity of φ and requires only the application of both rules for the main connective of φ. Now we can provide schemata of cut-free proofs for all schemata of axioms of some Hilbert formalization of classical propositional logic. Moreover, we also find that two-premise M-rules in this form are suitable for deriving troublesome theses (Peirce Law, e.t.c) without contraction. This is because now, due to the idempotency of multiset union, two-premise A-rules are simply special instances of more general M-rules. Hence contraction is not required for cut-free proofs of theses in contrast to ordinary M-rules. For example, cut-free proof of Peirce Law looks like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow p, q}\,(\Rightarrow W)}{\Rightarrow p, p \to q}\,(\Rightarrow\to) \qquad p \Rightarrow p}{(p \to q) \to p \Rightarrow p}\,(\to\Rightarrow *)}{\Rightarrow ((p \to q) \to p) \to p}\,(\Rightarrow\to)$$

Since primitive Cut allows simulation of Modus Ponens it follows that:

THEOREM 1. **CPLM** *provides adequate characterisation of classical propositional logic.*

At first it may seem a little advantage because one can avoid the problem of using contraction as primitive rule by taking two-premise A-rules instead, as in Ketonen's system. But, as we have seen in section 2, for such a calculus admissibility of contraction is required for the proof of cut admissibility. However, when we are using $(M - Cut*)$ even for Ketonen's version we do not need contraction in performing reductive steps (reducing cut-formula complexity) in cut admissibility proof. The respective figure from the example from section 2 now looks like that:

$$\dfrac{\Gamma \Rightarrow \Delta, \psi \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \psi, \Pi \Rightarrow \Sigma}{\psi, \Gamma\Pi \Rightarrow \Delta\Sigma}\,(Cut*)}{\Gamma\Pi \Rightarrow \Delta\Sigma}\,(Cut*)$$

5. Contraction revisited

It seems that in our system contraction is required neither for cut-free proofs of theses nor for proving cut elimination. But the last claim needs closer examination. Let us recall that Dragalin's proof of cut admissibility involves two kind of reductive steps: reduction of complexity of cut-formula and reduction of cut height. In his proof contraction is required only for transformations of the first type, as we have seen in section 2. In steps reducing cut height we have simply a permutation of cut with the preceding rule But we cannot be sure that the changes in two-premise rules (even if restricted to cut only) do not have an impact on reductions of cut height. In fact they have – as the two simple examples show. First, consider the following application of $(M - Cut*)$:

$$(M - Cut*)\,\dfrac{(\neg\Rightarrow)\,\dfrac{\Rightarrow p, q}{\neg q \Rightarrow p} \qquad p \Rightarrow q}{\neg q \Rightarrow q}$$

After standard transformation (i.e. pushing higher the application of cut in order to reduce height) we obtain:

$$(M - Cut*)\,\dfrac{\Rightarrow p, q \qquad p \Rightarrow q}{\Rightarrow q}$$

This is not problematic since we can use weakening and obtain required sequent. The second example is more problematic:

$$(M - Cut*)\,\dfrac{(\Rightarrow\neg)\,\dfrac{p \Rightarrow q}{\Rightarrow \neg p, q} \qquad q \Rightarrow \neg p}{\Rightarrow \neg p}$$

Now, after standard reduction step we obtain:

$$(\Rightarrow\neg)\,\dfrac{(M - Cut*)\,\dfrac{p \Rightarrow q \qquad q \Rightarrow \neg p}{p \Rightarrow \neg p}}{\Rightarrow \neg p, \neg p}$$

This is more problematic since in order to obtain required sequent we need contraction. At first sight it may seem hopeless but fortunately it is not. We still do not need contraction as primitive rule and we still do not need to prove its admissibility in a complicated way since by means of $(M - Cut*)$ we can prove its derivability (hence admissibility) in simple way:

$$(M - Cut*)\,\dfrac{\varphi \Rightarrow \varphi \qquad \varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

where left premise is by Lemma 1; similarly for $(\Rightarrow C)$. In fact, it is known that contraction is a special instance of $(A - Cut)$ with one premise axiomatic. But as we noted $(A - Cut)$ is a special case of $(M - Cut*)$ hence this result holds for our system as well. Derivability of contraction provides a solution to illustrated problem.

Before we present a (sketch of a) proof of cut elimination for our system let us examine carefully the sources of problems in the above two examples. Closer analysis shows that the first problem is due to the fact that side formula is also present in the second premise and after pushing cut up we loose one copy of this formula. In the second case, the principal formula is also present in the second premise and after pushing cut up and performing respective rule we obtain a duplication of this formula. We can summarize our observations and required additional steps in the following way:

1. If any side formula of the rule application which precedes cut is present in the second premise of cut and on the same side of a sequent, then after reduction step we need to apply weakening.
2. If the principal formula of the rule application which precedes cut is present in the second premise of cut and on the same side of a sequent, then after reduction step we need to apply contraction.

In order to provide general schemata of reduction of cut height we introduce the following notation:

Γ^{φ} is Γ with one copy of φ deleted if $\varphi \in \Gamma$; otherwise $\Gamma = \Gamma^{\varphi}$. $\Gamma^{\varphi,\psi}$ is the result of deletion of one copy of both formulae (if there are any).

6. Cut elimination

Now we can provide a simple proof of cut elimination for **CPLM**. It goes exactly like Dragalin's proof by double induction on cut-formula complexity and cut height but does not deserve complicated preliminaries like proving height-preserving invertibility of rules and height-preserving admissibility of weakening and contraction. The only preliminaries are Lemma 1 and the fact that contraction is derivable in the system - both with trivial proofs. Also we prove not admissibility of cut but its eliminability hence first we must prove the result for any proof where cut is applied once and then use (trivial) induction on the number of cut applications (as in Gentzen's original proof[5]).

In case if at least one premise is axiomatic or deduced by weakening on cut formula we just eliminate cut. This is trivial for the case of weakening, and for the case where one premise is axiomatic and the other premise has only one occurrence of cut formula. In case with more occurrences of cut formula it looks like that:

$$(M - Cut*)\ \frac{p \Rightarrow p \qquad p, p, \Gamma \Rightarrow \Delta}{p, \Gamma \Rightarrow \Delta}$$

Now the right premise of cut is not the same as the conclusion. Note however that at least one occurrence of p must be introduced by weakening in the proof of $p, p, \Gamma \Rightarrow \Delta$. Since it is not involved in any inferences we simple delete this application of weakening and all occurrences of p thus obtaining cut-free proof of $p, \Gamma \Rightarrow \Delta$.

Note that the problem of eliminating cut for this specific application of it, is the only reason for restricting axioms of **CPLM** to atoms.

In case of cut formula being principal in both premises we proceed exactly as in Dragalin's proof (by reduction of cut formula complexity) except that no contraction is needed after transformation. The case of $\varphi \wedge \psi$ looks like that:

$$(M - Cut*)\ \frac{(\Rightarrow \wedge *)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Pi \Rightarrow \Sigma, \psi}{\Gamma\Pi \Rightarrow \Delta\Sigma, \varphi \wedge \psi} \qquad \dfrac{\varphi, \psi, \Theta \Rightarrow \Lambda}{\varphi \wedge \psi, \Theta \Rightarrow \Lambda}\ (\wedge \Rightarrow)}{\Gamma\Pi\Theta \Rightarrow \Delta\Sigma\Lambda}$$

which is transformed into:

$$\frac{\Pi \Rightarrow \Sigma, \psi \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \psi, \Theta \Rightarrow \Lambda}{\psi, \Gamma^{\psi}\Theta \Rightarrow \Delta\Lambda}\ (M - Cut*)}{\Gamma^{\psi}\Pi\Theta \Rightarrow \Delta\Sigma\Lambda}\ (M - Cut*)$$

Now, if ψ is not in Γ we are done; otherwise we must additionaly apply weakening to restore this occurrence of ψ absorbed by multiset union of both antecedents in the first application of $(Cut*)$.

In case of cut formula not being principal in at least one premise we must reduce the height of cut taking into account possible deletions or

[5]Of course different strategies, like induction on cut-rank as in Schútte-Tait style proofs are also possible.

duplications of some formulae. Let us illustrate these reductions for one case of one-premise rule and one case of two-premise rule:

$$(M-Cut*)\,\dfrac{(\wedge\Rightarrow)\,\dfrac{\varphi,\psi,\Gamma\Rightarrow\Delta,\chi}{\varphi\wedge\psi,\Gamma\Rightarrow\Delta,\chi}\qquad \chi,\Pi\Rightarrow\Sigma}{\varphi\wedge\psi,\Gamma\Pi^{\varphi\wedge\psi}\Rightarrow\Delta\Sigma}$$

After standard transformation (i.e. pushing higher the application of cut in order to reduce height) we obtain:

$$(\wedge\Rightarrow)\,\dfrac{(M-Cut*)\,\dfrac{\varphi,\psi,\Gamma\Rightarrow\Delta,\chi\qquad \chi,\Pi\Rightarrow\Sigma}{\varphi,\psi,\Gamma\Pi^{\varphi,\psi}\Rightarrow\Delta\Sigma}}{\varphi\wedge\psi,\Gamma\Pi^{\varphi,\psi}\Rightarrow\Delta\Sigma}$$

Now, if Π is free of any ocurences of φ,ψ and $\varphi\wedge\psi$ we are done since both resulting sequents are identical. If any of φ or ψ belongs to Π, then we must additionaly apply weakening to the last sequent. If $\varphi\wedge\psi\in\Pi$, then we must additionaly apply contraction on $\varphi\wedge\psi$.

For two-premise rule we must distinguish between cases where cut-formula is only in one premise and harder one, where it is in both. We consider the latter.

$$(M-Cut*)\,\dfrac{(\Rightarrow\wedge*)\,\dfrac{\Gamma\Rightarrow\Delta,\varphi,\chi\qquad \Pi\Rightarrow\Sigma,\psi,\chi}{\Gamma\Pi\Rightarrow\Delta\Sigma,\varphi\wedge\psi,\chi}\qquad \chi,\Theta\Rightarrow\Lambda}{\Gamma\Pi\Theta\Rightarrow\Delta\Sigma\Lambda^{\varphi\wedge\psi},\varphi\wedge\psi}$$

after transformation we obtain:

$$(\Rightarrow\wedge*)\,\dfrac{(M-Cut*)\,\dfrac{\Gamma\Rightarrow\Delta,\varphi,\chi\qquad \chi,\Theta\Rightarrow\Lambda}{\Gamma\Theta\Rightarrow\Delta\Lambda^{\varphi},\varphi}\qquad \dfrac{\Pi\Rightarrow\Sigma,\psi,\chi\qquad \chi,\Theta\Rightarrow\Lambda}{\Pi\Theta\Rightarrow\Sigma\Lambda^{\psi},\psi}}{\Gamma\Pi\Theta\Rightarrow\Delta\Sigma\Lambda^{\varphi,\psi},\varphi\wedge\psi}$$

Again, if Λ is free of any ocurences of φ,ψ and $\varphi\wedge\psi$ we are done, in other cases we must additionaly apply weakening or contraction to the last sequent. All other cases of reduction of cut height, either on the proof of the left or on the proof of the right premise, look similar. Thus we obtain:

THEOREM 2. **CPLM** *admits cut elimination.*

As the sketch of the proof shows it it is as simple as Dragalin's proof but with significantly smaller set of necessary preliminaries. When constructing schemata of reduction cut complexity and cut height one must remember that everytime when some formula φ is displayed, except cut-formula, in the antecedent/succedent, we must change Γ from the antecedent/succedent of the second premise, into Γ^{φ}. This schematic notation covers cases where this φ is present also in the other premise (on the same side) and absorbed by multiset union of contexts.

7. Additive Cut

We have noted that two-premise A-rules are just special instances of our M-rules, so it seems that the results obtained by **CPLM** holds also for this special case. Moreover, such a system with all two-premise rules additive (including cut) may seem simpler because no new kind of reading the composition of contexts is required. However, the fact that $(A - Cut)$ is a special instance of $(M - Cut*)$ does not imply that conditions for its elimination are the same or even less restrictive; it is just the opposite. Cut elimination proof for such a system also requires height-preserving invertibility of all rules. It is needed not for proving admissibility of contraction, since this is derivable as we noted above. We must apply inversion in all cases of pushing cut applications upwards, in order to unify contexts of premises of new cuts. Let us illustrate this point:

$$(Cut)\ \dfrac{\varphi \wedge \psi, \Gamma \Rightarrow \Delta, \chi \qquad \dfrac{\chi, \varphi, \psi, \Gamma \Rightarrow \Delta}{\chi, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow)}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

is transformed into:

$$(Cut)\ \dfrac{\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta, \chi \qquad \chi, \varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi, \psi, \Gamma \Rightarrow \Delta}}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow)$$

where height-preserving invertibility for $(\wedge \Rightarrow)$ in the left premise is needed for unifying contexts.

Of course height-preserving invertibility again presupposes atomic axioms with contexts. In fact we need also weakening for reduction of cut complexity; here is an example:

$$(Cut)\ \dfrac{(\Rightarrow \wedge)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad \dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow)}{\Gamma \Rightarrow \Delta}$$

is transformed into:

$$\dfrac{\Gamma \Rightarrow \Delta, \psi \qquad \dfrac{\psi, \Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \psi, \Gamma \Rightarrow \Delta}{\psi, \Gamma \Rightarrow \Delta}\,(Cut)}{\Gamma \Rightarrow \Delta}\,(Cut)$$

where weakening (not necessarily height-preserving) is required to introduce ψ into the antecedent of $\Gamma \Rightarrow \Delta, \varphi$ since we need the same contexts in both premises of the first A-cut.

Summing up, such an A-system requires similar preliminaries for a proof of cut elimination like Ketonen's system for Dragalin's style proof. The only difference is that instead of proving in a complicated way height-preserving admissibility of contraction we obtain in a simple way its derivability.

8. Conclusion

Taking into account that proving invertibility of rules is important as such it may seem that advantages of our approach are not so big. But note that our solution applies also to extensions of classical logic with constants characterised by non-invertible rules. One such example is the familly of modal logics where typically rules for modal operators are non-invertible. It is an open question if some other advantages are not possible in richer contexts.

Summing up if we take ordinary multiset union as the way of combining contexts in two-premise rules we can obtain a system for classical logic which:

1. does not need contraction as primitive rule and does not require proving its admissibility by complicated induction;
2. does not require also other preliminary results like height-preserving admissibility of weakening and invertibility of rules;
3. allows for simple cut elimination proof;
4. may be applied also to stronger logics where no invertible sequent rules for constants are known.

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