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AN ARGUMENT FOR THE EXISTENCE OF GOD BY BOLZANO. A FORMALIZATION WITH A DISTINCTION BETWEEN *MENGE* AND *INBEGRIFF*

Abstract

Bernard Bolzano is the author of a text considered by him as an argument for the existence of God, originally published in the "Lehrbuch der Religionwissenschaft" (1834). His disquisition was formalized by H. Ganthaler and P. Simons in 1987 who based their approach on the well-known set theoretical interpretation of Bolzano's idea of multiplicity. The authors formulated a theory expressed in set theoretical language extended by some specific terms and constructed a proof as a formalization of the reasoning given by Bolzano. A problematic aspect of the proposal of Ganthaler and Simons is that it passes over the distinction of two different types of multiplicities which enable to point some interesting formal and material details of Bolzano's revised argumentation. In the frame of the presented proposal we invent two special meanings of Bolzano's notion of multiplicity: distributive and collective. In connection with this distinction we reconstruct the argument analyzed by Ganthaler and Simons and discuss the formal and philosophical power of the obtained formalization. The inspiration for our approach comes from the fact that Bolzano used in his text two terms *Menge* and *Inbegriff* in probably different meanings. However the main aim of this proposal is to take a new look at Bolzano's argument, rather than to provide historical investigations.

Keywords: formal ontology, theodicy, argument for existence of God, Bernard Bolzano

In 1987 H. Ganthaler and P. Simons formalized a text considered as the argument for the existence of God originally formulated by Bernard Bolzano and published in the "Lehrbuch der Religionwissenschaft" (LdR, first ed. 1834). It is surprising that this text has not been more widely discussed by other modern formal philosophers considering how well-known Bolzano is, thanks to his theological interests and works. This provides a good reason to undertake an analysis of the mentioned argumentation and to take into account the formalization of Ganthaler and Simons. Our aim will be to propose some reformulation of the attempt to describe Bolzano's disquisition by Ganthaler and Simons.

In introducing the main idea of the analyzed argumentation, let it be said that Bolzano was convinced that the crucial premise of Aristotelian-Thomistic argumentation for the impossibility of *regressum ad infinitum* in any chain of causes (or conditions) is false (cf. LdR §68, p. 3, 180) and therefore he intended to justify the thesis on the existence of God, which would be free of this statement. He understood God as *something real, which is unconditioned* (*ein Wirkliches, welches unbedingt ist*) and based his proof upon the assumption that the collection (*Inbegriff*) of all conditioned realities must be conditioned. Although the set (*Menge*)[1] of all conditioned realities may be infinite (*unendlich*), such a collection is also real and because it is conditioned it must have some real, external condition (*Bedingung*). This is because any real conditioned object must have real conditions. After all, this real object is outside of the considered set of all conditioned realities, therefore it must be unconditioned, and that is God.

Certainly, a formal representation of any text claiming to be adequate should be grounded in some preformal analysis of a richer context in which this text is formulated. Another, and of course main problem is how to find this ≪right≫ context. However Ganthaler and Simons made such an attempt. They referred to the well-known set theoretical interpretation of Bolzano's idea of multiplicity and they used set theoretical tools to formalize the considered argumentation. In effect they obtained a theory expressed in set theoretical language extended by some specific terms and constructed proof as a formalization of the reasoning given by Bolzano. In

[1] Following suggestion of Professor P. Rusnock the term *Menge* could be also translated as *number*. This translation also would legitimate our distributive interpretation presented in part 3.

our opinion a problematic aspect of the proposal of Ganthaler and Simons – further: *GS* theory – is that it passes over the distinction of two different types of multiplicities, which may be relevant to point out some interesting formal and material details in a revised argumentation of Bolzano. Bolzano uses in his proof two terms *Menge* and *Inbegriff* considered in general to be especially ambiguous in his philosophy and mathematics.[2] We are going to maintain this plurality at least in the sense that in our formalization we will be able to speak simultaneously about two different types of multiplicities that are considered in modern literature as known and plausible interpretations of Bolzano's ideas – about both distributive and collective sets (mereological sums).[3] In the frame of this proposal we will reconstruct the argument analyzed by Ganthaler and Simons.

We base our theory on a certain fragment of the system ZFM – an extension of the Zermelo-Fraenkel set theory by S. Leśniewski's mereological axioms – which was originally described and elaborated upon in (Pietruszczak, 2000). This choice may be also grounded in other circumstances: the Zermelo-Fraenkel approach codes Cantor's notions of the distributive set and actual infinity and these are found in Bolzano's lecture.[4]

1. The sources. Texts from LdR

To point out some circumstances which may be of importance for the content of LdR to be considered as not attractive enough for investigation, let us note that the first edition of LdR was published in 1834 by anonymous students of Bolzano, who described it as a collection of lecture notes. On the title page of the specimen copy of LdR we can read: *Lehrbuch der Religionswissenschaft, ein Abdruck der Vorlesungshefte eines ehemaligen Religionslehrers an einer katholischen Universität von einigen seiner Schüler gesammelt und herausgegeben.* (Textbook of Religious Studies, A printing of Lecture Notes of a Former Teacher of Religion at a Catholic University

[2]A rich list of different meanings which may be linked with *Inbegriff* is to be found in (Simons, 1997).

[3]The popular idea of set theoretical interpretation of multiplicity considered by Bolzano especially in (Bolzano, 1851) is the subject of many discussions, which are mentioned with detailed references in (Krickel, 1995). In the same book an analysis and formalization of the mereological interpretation of Bolzano's *Inbegriff* is also proposed.

[4]These associations, well known in the philosophy of mathematics, are commented in for example (Murawski, 2012).

collected and published by some of his students). So, the original text was not signed by Bolzano, although he is described as an author in the preface and today the text of LdR is considered as his own disquisition.[5]

We quote the analyzed fragment of LdR in its modern transcription:

§. 67. Daseyn Gottes

[...]

a) Es gibt doch überhaupt einiges Wirkliche. Dieses mein eigenes Urtheil, als eine Erkenntniß, oder als ein Gedanke betrachtet, ist schon selbst etwas Wirkliches.

b) Ich habe nun irgend Eines von diesen Wirklichen, z. B. das Wirkliche A, heraus, und frage denjenigen, der and dem Daseyn eines unbedingten Wirklichen noch zweifelt, wofür er das Wirkliche A erklären wolle; ob für unbedingt, oder nicht? Thut er das Erstere: so gestehet er selbst, daß es ein unbedingt Wirkliches, d. h. einen Gott gebe.

c) Will er das Wirkliche A nicht für unbedingt erkären: so verlange ich, daß er gleich alle die wirklichen Dinge A, B, C, ... welche nicht unbedingt sind, in einen Inbegriff zusammenfasse. Dieses muß wenigstens im Gedanken möglich seyn, gesetzt auch, daß ihre Menge unendlich wäre.

d) Ich frage nun ferner, wofür er diesen Inbegriff aller bedingten Wirklichkeit, der gewiß selbst auch etwas Wirkliches ist, erkläre, ob für ein unbedingtes oder bedingtes Wirkliche: Thut er das Erstere, so gesteht er uns abermals ein unbedingtes Wirkliche zu.

e) Im widrigen Falle erinnere ich, daß jedes Wirkliche, welches nicht unbedingt ist, das Daseyn eines andern Wirklichen voraussetzt, durch welches es bedingt ist. Auch der Inbegriff aller bedingten Wirklichkeit also setzt noch ein anderes Wirkliche, durch welches er eben bedingt ist, voraus.

f) Dieß andere Wirkliche nun muß etwas Unbedingtes seyn; denn wäre es bedingt: so würde es eben darum mit zu dem Inbegriff aller bedingten Wirklichkeit gehören.

g) So gibt es denn also in einem jeden Falle ein Wirkliches, welches unbedingt ist, d.h. einen Gott.

Anmerkung. Die Annahme d, daß der Inbegriff aller bedingten Wirklichkeit selbst etwas Unbedingtes wäre, gestatte ich dem Gegner nur vor der Hand, weil ich es zum Beweise des gegenwärtigen Satzes nicht nöthig habe, sie zu widerlegen. In der Folge werde ich gleichwohl die Unrichtigkeit dieser Behauptung, wenn unter diesen Dingen Substanzen

[5] This whole situation was probably connected with the fact that Bolzano's entitlements to teach and publish were restricted by the Church since 1819/20. Also the general concept of theology by Bolzano was considered as so different to Catholic ideas that LdR was included on the Index from 1839.

verstanden werden sollen, zeigen. (§ 67, 205-207, in or. 177-179)

We take the following translation:

§. 67. Existence of God

[...]

a) There is at least something real (*Wirkliche*). This, my own judgment, considered as knowledge or as an idea (*Gedanke*), is itself already something real.

b) Now I choose one of these real things (*diesen Wirklichen*), let's say, real A, and I ask someone, who doubts the existence of some unconditioned real thing, if this A is unconditioned or not? If he agrees with the first case, he admits that there is an unconditioned real thing, i.e. God.

c) If he does not want to declare that the real A is unconditioned, then I ask him to put together all real things (*Dinge*) A, B, C, ... which are not unconditioned in one collection (*Inbegriff*). This must be at least possible in thought, even in the case that this set (*Menge*) is infinite.[6]

d) I ask him furthermore, whether he believes that this collection (*Inbegriff*) of all conditioned reality, which is itself real, is an unconditioned or conditioned real thing. If he chooses the first then again he admits that there is some unconditioned real.

e) On the other hand, I remember, that every real, which is not unconditioned supposes another real by which it is conditioned. Also the collection (*Inbegriff*) of all conditioned reality supposes another real by which it is conditioned.

f) This reality must be something unconditioned, for if it were conditioned, it would belong to the collection (*Inbegriff*) of all conditioned reality.

g) So there is in each case some real which is unconditioned, i.e. God.

Remark. Supposition d, that the collection (*Inbegriff*) of all conditioned reality (*Wirklichkeit*) is itself unconditioned I admit only for a moment because I don't need to refute it for the purpose of proving the present theorem. Later on I will prove that this claim does not hold if things are understood as substances. This will be proven later.

For clarity of our considerations, we use the English term *set* for the German term *Menge* and *collection* for *Inbegriff* without taking in advance already some defined, precise meanings.

[6] cf. 2.

2. Formalism of Ganthaler and Simons. Report and remarks

The vocabulary of GS language consists of: (i) individual variables: x, y, z, $\ldots$ (IV), (ii) individual constant b, (iii) identity predicate $=$ and predicate ϵ, (iv) primitive specific predicates: $W\ldots$ read ... *is real* (*wirklich*) and $\ldots R\ldots$ read ... *is a condition of* ... (*bedingt*), (v) logical symbols: abstraction operator $\{...:...\}$, truth connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$, quantifiers: $\exists, \forall$ and (vi) brackets.[7]

Terms (Γ_{GS}) and formulae (FOR_{GS}) are built in a standard way, so:
(i) if $v \in IV$ or $v \equiv b$ then $v \in \Gamma_{GS}$; (ii) if $v \in IV$ and $A \in FOR_{GS}$ then $\{v\colon A\} \in \Gamma_{GS}$; (iii) if $v, v' \in \Gamma_{GS}$ then: $v = v'$, $v \,\epsilon\, v', Wv, vRv' \in FOR_{GS}$; (iv) if $A, B \in FOR_{GS}$ and $v \in IV$ then $\neg A, A * B$ $(for * \in \{\wedge, \vee, \rightarrow, \leftrightarrow\})$, $\forall_v A, \exists_v A \in FOR_{GS}$.

The authors take following metadefinitions:

Def$B$. $Bx \equiv \exists y(yRx)$ (x is conditioned (*bedingt*))
Def$b$. $b = \{x\colon Wx \wedge Bx\}$ (set of all conditioned reals)
Def$E!$. $E!x \equiv \exists y(y = x)$ (x exists)

and they list two groups of axioms called:

- *set theoretical axioms*

V1. $\exists_x Fx \rightarrow E!\{x\colon Fx\}$
(If there is some x such that Fx, then the set of all x such that Fx exists)
V2. $E!\{x\colon Fx\} \rightarrow \forall_y(y\epsilon\{x\colon Fx\} \leftrightarrow Fy)$
(Comprehension schema restricted to existent sets)

- *premises*

P1. $\exists_x Wx$
(There is something real)
P2. $E!\{x\colon Fx\} \wedge \forall_x(Fx \rightarrow Wx) \rightarrow W\{x\colon Fx\}$
(If the set of all x that are F exists and every element of it is real then this set is also real)

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[7] To avoid needlessly rich notation we use symbols: $=, \{...\}$ also in metalanguage. The right meanings will be seen from the context.

P3. $\forall_x \forall_y (Wx \land yRx \to Wy)$
(If x is conditioned by y and x is real then y is real)
P4. $\forall_z (\forall_x (x \epsilon z \to \forall_y (yRx \to y \epsilon z)) \to \neg Bz)$
(If all elements of z are conditioned only by elements of z then z is not conditioned)

The authors consider definitions Def$b$ and Def$B$ as formalizations of fragments from (c) and (e) respectively. Next they also point out these fragments, which are represented by premises P1, P2 and P3. Axiom P1 is mentioned as a formalization of the premise from (a), P2 has to express the content of part (d) and P3 is intended to be a formalization of a premise from (e). P4 – here called *the foundation principle* – is said not to correspond to any part of the quoted text.

Let us formulate the main thesis of Ganthaler and Simons' formalization:

T*. $\exists_x (Wx \land \neg Bx)$
(There exists an object which is real and not conditioned)

which is proved using dilemma and this seems to maintain the course of Bolzano's argumentation. The derivation is correct and we will keep a similar structure in our further proposal.

Let us now come back to V1 and V2 and say that in connection with the unclear formal background of the presented approach, they actually lead to some difficulties.

To see the first complication we take V1 and DefE! to get:

$E!\{x \colon \exists_z (z = x)\}$.

Now we apply V2 and we get $y \epsilon \{x \colon \exists_z (z = x)\} \leftrightarrow \exists_z (y = z)$ and so, we obtain also $\exists_y (y = \{x \colon \exists_z (z = x)\})$ which expresses the existence of the universal set which leads in connection with power set axiom to the well known Cantor's antinomy.

The second problem is connected with P2 and again with V1 and V2. Because of P1 and V1 we get $E!\{x \colon Wx\}$ and directly from P2 we obtain $W\{x \colon Wx\}$. So, by V2 we also have $\{x \colon Wx\} \epsilon \{x \colon Wx\}$ which contradicts $\forall_x \neg (x \epsilon x)$ - a consequence of the regularity axiom.

The above facts show that it is impossible to base the formalization of Ganthaler and Simons on any set theoretical system that excludes the existence of universal set or accepts the axiom of regularity. However even if somebody could take as a formal frame some poorer fragment of set theory, he would not get rid of the problem of the unclear sense of V1, V2 and P2 axioms which are involved in the sketched derivations. After all P4 is also problematic.

As we have already noted, indeed axioms V1 and V2 have no counterparts in Bolzano's argument and it even seems that there is no fragment of the considered text in which a term somehow ≪associated≫ with predicate E! could be found. Authors assume V1 as some weaker version of the idea mentioned by Bolzano from (Bolzano, 1851):

> Bolzano [...] spricht erst dann von einer Menge bzw. einem Inbegriff, wenn diese(r) mindestens zwei Elemente hat. (Ganthaler, Simons, 1987, 471)

> Bolzano [...] only speaks about sets, resp. collections if they contain at least two elements.

But in such a context V1 is even more problematic, because the quoted fragment suggests not V1 and at most the converse implication to V1. But deleting V1 breaks down the presented proof of the thesis T*.

To consider V2, it is not clear how to understand abstract terms. If we apply the standard rules for eliminating them[8] we simply get a logically valid formula.

At the end we should supplement the list of problems connected with the proposal of Ganthaler and Simons by some philosophical difficulties which can also be seen in formulation of P2 and P4.

Axiom P2 links predicates $E!$ and W. The grammatical possibility of applying predicate W both to individual and abstract terms has the following effect in P2: in the context of the considered formalism, the reality of individuals (more precisely: objects that are not sets) is not distinguishable from the reality of sets. Such a construction seems to also miss the intentions of Bolzano. Although he regards the notion of *being real* as a primitive one (this we will consider also in 3), he only considers *substances* and *adherences* as real objects. This idea is explicitly expressed just after textually analyzed argumentation:

> 1. [...], daß alles W i r k l i c h e zu einer von folgenden zwei Arten gehöre, daß es entweder S u b s t a n z oder A d h ä r e n z sey, daß Adhärenz bloß

[8] cf. e.g. (Takeuti, Zaring, 1982, Def. 4.2).

> ein solches Wirkliche heiße, das sich an einem Andern als eine Beschaffenheit dasselben befindet. Was sich n i c h t an einem Andern befindet, sondern, wie man dieß auszudrücken pflegt, f ü r s i c h b e s t e h e t, das heiß S u b s t a n z oder W e s e n. [...], daß alles, was anfängt oder aufhört, nur eine Adhärenz sey, während jede Substanz etwas Beständiges ist, das weder anfangen, noch vergehen kann, sondern, so ferne es einmal ist, zu aller Zeit seyn muß . (LdR, /S 70. Substanzialität Gottes, 185)
>
> 1. that all r e a l belong to one of the following two kinds, that it is s u b s t a n c e or a d h e r e n c e, that adherence is called a real, which is attached to another thing as property. That which is not attached to another, but, as one likes to express, exists by i t s e l f, is called s u b - s t a n c e or e s s e n c e, [...], that everything that begins or ends is only an adherence, whereas every substance is something stable which cannot begin or end but, insofar as it exists once, has to be forever.

Actually substances and adherences are objects of different kinds from sets in sense of modern set theoretical systems.[9]

The other problematic question is how to set the interpretation of the predicate B with the meaning of *being conditioned* intended by Bolzano. Let us at least notice that from P4 just by classical logic we obtain the implication $\forall_z (Bz \rightarrow \exists_x \, x\epsilon z)$ and so, in the frame of GS theory every conditioned object must be a set while Bolzano explicitly considers in his argumentation a concrete thing (*Dinge*) A which may be conditioned and this comprises the main and essential part of his proof.

3. A formalization of Bolzano's argument with two kinds of multiplicities

Let us now propose a formalization of Bolzano's argument inspired by the approach of Ganthaler and Simons but free of the difficulties mentioned in 2.

The first essential difference is that now we keep the distinction of two different kinds of multiplicities: sets and collections.

We use the language with the following vocabulary: (i) individual variables: $x, y, z, \ldots$ (IV), (ii) individual constant b, (iii) identity predicate $=$, predicate ϵ, one place predicate Z read: *is a set*, (iv) two place mereological

[9] Some formal description of these notions is presented in (Krickel, 1995) and in the context of this approach they are linked with the collective interpretation of *Inbegriff*.

predicate $\sqsubset$ read: *is a part of*; (v) primitive specific one place predicates: W, B, and two place predicate R read as before, (vi) logical symbols: truth connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$, quantifiers: $\exists, \forall$ and (vii) brackets.

The set of terms consists of elements of IV together with b. Formulae are built up in the usual way. We do not use abstract terms and we use formulae with predicates Z and $\sqsubset$.

We will take as a formal basis of the proposed theory the system ZFM which is as we already mentioned - an extension of Zermelo-Fraenkel set theory by mereological axioms formulated in (Pietruszczak, 2000).

ZFM is characterized by the following axioms and definitions:

- logical valid formulas of first order logic with identity ($PL1 =$)
- Zermelo-Fraenkel axioms for Z, ϵ in formulation from (Pietruszczak 2000, 172-181) (ZF)
- axioms for $\sqsubset$ and definitions (MA):
  (λ1) $\forall_x \forall_y (x \sqsubset y \rightarrow \neg(y \sqsubset x))$
  (Asymmetry of part relation)
  (λ2) $\forall_x \forall_y \forall_z (x \sqsubset y \wedge y \sqsubset z \rightarrow x \sqsubset z)$
  (Transitivity of part relation)
  ($\sqsubseteq$) $x \sqsubseteq y \leftrightarrow x \sqsubset y \vee x = y$
  (Ingrediens)
  ($\circ$) $x \circ y \leftrightarrow \exists_z (z \sqsubseteq x \wedge z \sqsubseteq y)$
  (Overlapping)
  ($\wr$) $x \wr y \leftrightarrow \neg(x \circ y)$
  (Disjointness)
  (σ) $x \sigma z \leftrightarrow Zz \wedge \forall_y (y \epsilon z \rightarrow y \sqsubseteq x) \wedge \forall_y (y \sqsubseteq x \rightarrow \exists_u (u \epsilon z \wedge u \circ y))$
  (Mereological sum)
  (λ3) $\forall_x \forall_y \forall_z (x \sigma z \wedge y \sigma z \rightarrow x = y)$
  (Extensionality)
  (λ4) $\forall_z (\exists_y (y \epsilon z) \rightarrow \exists_x (x \sigma z))$
  (There are only mereological sums of something)
  (α) $\forall_z (Zz \rightarrow \neg \exists_x (x \sqsubset z))$
  (Specific axiom for ZFM: Sets are not collections)

To obtain our theory – called in the following BB – we add the following definition and specific axioms:

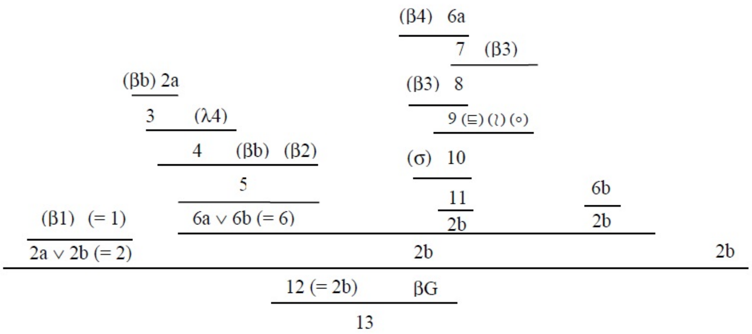
(βG) $\forall_x(Gx \leftrightarrow Wx \wedge \neg Bx)$
(x is God)
(βb) $Zb \wedge \forall_x(x\epsilon b \leftrightarrow Wx \wedge Bx)$
(The object consisting of all real and conditioned things is a set)
($\beta 1$) $\exists_x Wx$
(There is something real, cf. P1)
($\beta 2$) $\forall_v \forall_y(v\sigma y \wedge \forall_z(z\epsilon y \to Wz) \to Wv)$
(Each mereological sum of real objects is real)
($\beta 3$) $\forall_x \forall_y(Wx \wedge yRx \to Wy)$
(Conditions of real objects are real, cf. P3)
($\beta 4$) $\forall_x(Bx \to \exists y(yRx \wedge (y \wr x \vee \exists_z(z \wr x \wedge zRy))))$
(If x is conditioned, then x has some condition which is disjoint with x or this condition is conditioned by an object which is also disjoint with x)

The main thesis in our formalization is:

T*' $\exists_x Gx$

1. $\exists_x Wx$	$(\beta 1)$
2. $\exists_x(Wx \wedge Bx) \vee \exists_x(Wx \wedge \neg Bx)$	$(PL1 =)/(= 2a \vee 2b)$
$2a$. $\exists_x(Wx \wedge Bx)$	*Assumption*
3. $\exists_x(x\epsilon b)$	$(\beta b), 2a$
4. $\exists_v(v\sigma b)$	$(\lambda 4), 3$
5. $\exists_v(v\sigma b \wedge Wv)$	$(\beta b), (\beta 2), 4$
6. $\exists_v(v\sigma b \wedge Wv \wedge Bv) \vee \exists_v(v\sigma b \wedge Wv \wedge \neg Bv)$	$5/(= 6a \vee 6b)$
$6a$. $\exists_v(v\sigma b \wedge Wv \wedge Bv)$	*Assumption*
7. $\exists_v(v\sigma b \wedge Wv \wedge \exists_y(yRv \wedge (y \wr v \vee \exists_z(z \wr v \wedge zRy))))$	$6a, (\beta 4)$
8. $\exists_v(v\sigma b \wedge (\exists_y(Wy \wedge y \wr v) \vee \exists_y(Wy \wedge \exists_z(z \wr v \wedge zRy))))$	$7, (\beta 3)$
9. $\exists_v(v\sigma b \wedge (\exists_y(Wy \wedge y \wr v) \vee \exists_z(Wz \wedge z \wr v)))$	$8, (\beta 3)$
10. $\exists_v(v\sigma b \wedge \exists_y(Wy \wedge \neg(y \sqsubseteq v))$	$9, (\sqsubseteq), (\wr), (\circ)$
11. $\exists_y(Wy \wedge \neg(y\epsilon b))$	$10, (\sigma)$
12. $\exists_y(Wy \wedge \neg By)$	$(\beta b), 11, 12 = 2b$
13. $\exists_x Gx$	$12, (\beta G)$

To see the structure of the proof we draw the following derivation tree:



As can be seen, the proof of T$^{*\prime}$ requires actually only a weak fragment of ZFM and so we could say that at least this part of the proposed version of Bolzano's theodicy does not refer to any especially powerful assumptions about either sets and collections. Actually set theoretical background comes down to the use of set theoretical language. On the mereological level we need only definitions of mereological sum (σ), ingrediens ($\sqsubseteq$), disjointness ($\wr$), overlapping ($\circ$) and axiom ($\lambda 4$). Axiom ($\lambda 4$) links the relation of being element with the notion of mereological sum (and so with that of part relation) restricting the possibility of constructing mereological sums only from non empty sets. As we see, the formalized argumentation did not engage in particular axioms about asymmetry and transitivity of parthood relation[10] as well the extensionality axiom ($\lambda 3$) which guarantees together with ($\lambda 4$) the uniqueness of the mereological sum of elements of any set.[11]

To note a philosopically interesting nerve of Bolzano's approach we may see that the inference from the line $2a$ till the line 12 justifies the implication

(PT) $\quad \exists_x(Wx \wedge Bx) \rightarrow \exists_x(Wx \wedge \neg Bx)$

(If there exists something real and conditioned, then there exists something real and unconditioned (i.e. God).)

(PT) could be used as lemma in a simpler proof of T*' (we would need

[10]Bolzano wouldn't accept the transitivity of his *part* relation.

[11]We do not use an individual constant for the mereological sum of all real and conditioned objects.

only $\beta 1$). In our proposal we keep the original structure of Bolzano's argumentation.

In commenting the specific notions of BB let us say, that similar to Ganthaler and Simons, we follow the intention of Bolzano and adopt the notion of *being real* (W) as a primitive one:

> 5. Der Begriff des W i r k l i c h e n braucht keine weitere Erläuterung; wohl aber jener des Unbedingten. Unbedingt heißt mir dasjenige, was keine Bedingung hat. (LdR, § 66 *Begriff Gottes*, 175)
>
> 5. The notion of being r e a l does not require any further explanation; although this is not the same with being unconditioned. Unconditioned means to me that, which has no condition.

Concerning the notion of *being conditioned*, we do not assume definition Def$B$ and we note the relational nature of B in ($\beta 4$) (we link B with R). Actually Def$B$ is also not needed in derivation of T* and it seems to be some kind of over-interpretation of the fragment (e) taken in GS.[12] Assuming that *to not be unconditioned* should be understood as equivalent to *to be conditioned*, we may say that in (e) at most a necessary condition *to be conditioned* is formulated. Moreover: Bolzano says in (e) that every conditioned real supposes **another** real by which it is conditioned and in the fragment of the proof of T* referred to (e), authors of GS use P3 together with some equivalent of P4 and not Def$B$. Anyway, we partly follow GS taking as an axiom ($\beta 3$) which is the same as P3 here we use the relation of *being conditioned by* (R) and we accept that an attribute of *being real* is inherited by the converse of this relation. Then we complete our interpretation of (e) in a different way as it is in GS: we accept the implication ($\beta 4$) which expresses the mentioned necessary condition *to be conditioned* and consider the distinction of a conditioned object from its condition by means of mereological disjointness ($\wr$). In the same way we would interpret the following explanation by Bolzano:

> 7. [...] denn es ist überhaupt ungereimt, daß etwas Grund von sich selbst seyn könne, sondern der Grund und seine Folge, und eben so die Bedingung und das Bedingte sind immer zwei von einender verschiedene Gegenstände. Mann sollte also, statt zu sagen, daß der Grund von Gottes Daseyn in ihm selbst liege, eigentlich sagen, daß es gar keinen Grund, ja nich einmal eine Bedingung seines Daseyns gebe. (LdR, § 66 *Begriff Gottes*, 177)
>
> 7. [] because it is generally absurd, that something could be reason for itself, but reason and it's effect, like condition and conditioned, are always

[12] Even if the equivalence Def$B$ may be suggested in other texts by Bolzano.

> two different things.Therefore instead of saying that the reason for God's existence lies in itself one should say that there is no reason at all for God's existence, even nor a condition for His existence.

So we claim that if something is conditioned then it has at least intermediately external conditions for its existence. (The weakening of this assumption to speak about *being conditioned* only as to be dependent upon the existence of something different passes over the asymmetry between any conditioned complex object and some of its conditions, namely its proper parts: in this case all parts are conditions of a whole to the same extent as the whole is the condition of its parts.) [13]

Like Ganthaler and Simons we take axiom ($\beta 1$) which is the same as P1 referred to the content of (a). Equivalence (βG) may be added to the GS as a definitional extension of it. We take the same understanding of God as a real and unconditioned object. After all, such a definition is repeated by Bolzano both in the considered argument and in other parts of his "Natürliche Dogmatik" from LdR (c. f. LdR, § 66, p. 6, p. 7; §70, p. 2)

The possibility of distinguishing sets from mereological sums gives some advantage to our approach. At first let us recall the already commented ($\beta 4$) in contrast to P4. In GS we will easily get from P4 the unexpected conclusion that every condition object has to be a set and this is not a consequence ($\beta 4$). The other advantage of our approach may be seen in connection to ($\beta 2$) – a formalization of (d) in contrast to P2 from GS. In our case the reality of the multitude of realities is grounded on the fact that such an object is just a mereological sum – let us say: it is an object of the same *category* as its parts – and not a set that ≪exists≫ in an unclear sense of predicate $E!$.

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[13] The concept of dependency of parts on wholes is used in (Nieznański, 1983) to formalize Leibniz's argumentation for the existence of *ratio sufficientis essendi*. If we would assume in BB additionally that to be conditioned in sense of R is transitive: $\forall_{x,y,z}(xRy \wedge yRz \rightarrow xRz)$ then it would come out that our notion of being conditioned in sense of B is stronger than its counterpart of Nieznański – in our notation we could formulate a definition: $B_N x \leftrightarrow \exists_y(yRx \wedge \neg(y \sqsubseteq x))$. To strengthen our notion of being unconditioned in sense $\neg Bx$ - just to exclude the situation in which an unconditioned object could have some conditions which overlap with it (and this is excluded in case of $\neg B_N x$) – we would have to additionally assume that such an object is simple in a sense that it does not have proper parts. Actually we will come back to this problem in our last remarks.

Despite some benefits of BB versus GS, we should however at least comment upon the problem of the philosophical power of BB itself and so upon the proof of T$^{*\prime}$. To see the ≪real≫ (and weak) content of the specific axioms of BB let us only sketch the following interpretations of them.

We consider a model only for the fragment of BB characterized by specific axioms βG, βb, $\beta 1 - \beta 4$, mereological axioms $\lambda 1 - \lambda 4$, axiom (α) and two set theoretical axioms

(ext) $\forall_{x,y}(Zx \wedge Zy \wedge \forall_z(z\epsilon x \leftrightarrow z\epsilon y) \rightarrow x = y)$

$(\exists\varnothing)$ $\exists_x(Zx \wedge \neg\exists_y\ y\epsilon x)$

We take the universe $U = \{\mathbf{b}, \mathbf{g}\}$, where $\mathbf{b} \neq \mathbf{g}$. Object $\mathbf{b}$ is considered as the interpretation of constant b. $\mathbf{W} = \{\mathbf{g}\}$, $\mathbf{Z} = \{\mathbf{b}\}$, $\mathbf{B} = \emptyset$ are interpretation of predicates: W, Z, B respectively. We assume that the interpretation of predicates: ϵ, $\sqsubset$, R is: $\in^* = \sqsubset^* = R^* = \emptyset$.
We may easily see that axioms $\lambda 1 - \lambda 4$ as well as $\beta 2 - \beta 4$ are true just because for every case implications from them have false antecedenses. Axiom $\beta 1$ is true because $\mathbf{W} \neq \emptyset$. βb is also true since $\mathbf{b} \in \mathbf{Z}$. Because $\mathbf{B} = \emptyset$, $\mathbf{b}$ is empty set in the sense of $(\exists\varnothing)$. According to our interpretation $\mathbf{b}$ is the only one set (in the sense of predicate Z), so (α) is also true.
To see that T*' is true we may use (PT) which is true again because of falsity of antecedens and $\beta 1$, which is also true since $\mathbf{g} \in \mathbf{W}$. By βG we see that $\mathbf{g}$ is a God-like being.

To have such a model of the mentioned BB fragment in which the set of all real and conditioned objects is not empty let us take the universe $U = \{\mathbf{b}, \mathbf{g}, \mathbf{r}, \varnothing\}$, where $\mathbf{b} \neq \mathbf{g} \neq \mathbf{r} \neq \varnothing$. $\mathbf{W} = \{\mathbf{g}, \mathbf{r}\}$, $\mathbf{B} = \{\mathbf{r}\}$, $\mathbf{Z} = \{\mathbf{b}, \varnothing\}$, $\in^* = \{< \mathbf{r}, \mathbf{b} >\}$, $\sqsubset^* = \emptyset$, $R^* = \{< \mathbf{g}, \mathbf{r} >\}$. The object $\mathbf{b}$ - the interpretation of constant b - is the set with only one real and conditioned object $\mathbf{r}$. The existence of $\varnothing$ follows from $(\exists\varnothing)$.
Axioms $\lambda 1, \lambda 2$, (α) are true in view of the fact that $\sqsubset^* = \emptyset$. $\lambda 3, \lambda 4$ are also true: there are only two sets in the sense of Z: $\mathbf{b}$, $\varnothing$ and each of them satisfies the implications in $\lambda 3$ and $\lambda 4$. Our interpretation provide the truth of βb and $\beta 1 - \beta 4$. According to βG we can see that $\mathbf{g}$ is a God-like object.

We put our formalism in a powerful frame of system ZFM of Professor Pietruszczak and this we should take in account in an interpretation of

BB theory.[14] Of course we did not need it really for the construction of our rather plain models, but it is probably the richest possible context in which one could consider our argument as a part of a more complex theological system.

After all let us also note that our simple interpretations may hide some material weakeness of Bolzano's argument. In both proposed models a God-like object is an individual in the following sense:

$$(I) \qquad Ix \leftrightarrow \neg Zx \wedge \forall_y (y \sqsubset x \rightarrow \neg Zx)$$

as well a mereological atom:

$$(AT) \qquad Atx \leftrightarrow \forall_y (y \sqsubseteq x \rightarrow x = y)$$

but this is not forced by the considered specific axioms.
If we change e.g. our first interpretation and we add $\mathbf{g}$ to the set $\mathbf{Z}$ (all other details would not be changed) our axioms still are true. [15] (Here $\mathbf{g}$ is still a mereological atom (cf. AT and (α).)
To consider the situation in which a God-like object is not a merelogical atom we could take the universe $U = \{\mathbf{g}, \mathbf{g}', \mathbf{b}\}$ $(\mathbf{g} \neq \mathbf{g}' \neq \mathbf{b})$, $\mathbf{W} = \{\mathbf{g}, \mathbf{g}'\}$, $\mathbf{Z} = \{\mathbf{b}\}, R^* = \in^* = \mathbf{B} = \emptyset$ and $\sqsubset^* = \{< \mathbf{g}, \mathbf{g}' >\}$. We see that $\mathbf{b}$ is the empty set in the sense of $(\exists\varnothing)$. Again it is easy to check that axioms $\lambda 1 - \lambda 4$, (α) and βb, $\beta 1 - \beta 4$ are true. In this case we have two real and unconditioned objects: $\mathbf{g}$ and $\mathbf{g}'$, where $\mathbf{g}'$ is not a mereological atom.
Bolzano was probably aware of some lack in his argument because of passing over the fact that God as a real and unconditioned object should also have some sort of *individual status*. In the frame of his philosophy this individuality was identified with already mentioned substantiality which is actually one of the essential attributes of God. This idea, expressed in *Anmerkung*, was also elaborated in the anounced text:

> 2. Daß nun Gott e i n e S u b s t a n z s e y, ist abermals eine Behauptung, die man den ausgemachtesten Lehrsätzen der natürlichen Religion des Menschengeschlechtes beizählen kann; denn sicher hat noch Niemand, der das Daseyn eines unbedingt Wirklichen zugab, daran gezweifelt, daßdieses

[14] In this case we could probably proceed in a similar way as it was proposed in (Pietruszczak, 1996). Assuming to have a model of ZF (with distributive atoms) we would start with adding to its universe individuals from any of our model (in the sense of the next definition of I). After forming all possible mereological sums from all considered objects we would form all distributive sets of all objects constructed until now and then repeat this procedure *ad infinitum*.

[15] By the way let us note that although mereological sums of distributive sets of reals are real (cf. $\beta 2$), it is left open if there is any real distributive set.

> Wirkliche eine Substanz seyn müsse.
>
> 3. Auch der G r u n d, auf dem diese Wahrheit beruht, ist wohl leicht einzusehen. Denn jede A d h ä r e n z setzet das Daseyn einer Substanz, an der sie sich befindet, als eine Bedingung zu ihrem eigenen Daseyn voraus. Das unbedingt Wirkliche kann also keine Adhärenz seyn, und folglich bleibt nichts übrig, als daß es eine Substanz sey. (§70, 183)
>
> 2. that God now is a s u b s t a n c e is again a claim which can be counted as one of the well known theorems of the natural religion of mankind; that certainly nobody who admitted the existence of an unconditioned real doubted that this real must be a substance.
>
> 3. Also the r e a s o n upon which this truth rests is easily discerned. Since each a d h e r e n c e presupposes a substance to which it is attached as a condition for its own existence. The uncondioned real can therefore not be an adherence and so it must be a substance.

The problem of linking Bolzano's notion of *Substanz* with the above introduced notions of individual and mereological atom we leave here open. However any possibly justified solution should enable to undertake the next crucial issue how to incorporate substaniality of God-like being in the analyzed argument. Otherwise we would say that it does not concerns the existence of God even in some weak sense of classical theodicy.

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