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A NOTE ON THE ADEQUACY OF JERZY KALINOWSKI'S $\mathbf{K}_1$ LOGIC

Abstract

Jerzy Kalinowski's $\mathbf{K}_1$ logic is one of the first systems of deontic logic. Kalinowski presented it in two forms: as an axiomatic system and with the use of deontic tables analogous to Łukasiewicz's thee-valued propositional logic. Adequacy of those two approaches is proven.

Introduction

The $\mathbf{K}_1$ system was one of the world's first deontic logics. It was published in a journal article [4] in 1953, but earlier, in 1951 presented in Kalinowski's habilitation dissertation (see also the description of the system in Kaliniowski's later book [5]). The importance of that contribution to the development of deontic logic was stressed by G.H. von Wright who listed Kalinowski, along with O. Becker and himself, as one of the three 'founding fathers' of modern deontic logic [10]. The trivalent semantics for deontic logic introduced for $\mathbf{K}_1$ was later used in [2, 1] and recently in [6, 8, 7].

Kalinowski presented his system in two forms: as an axiomatic system and with the use of deontic tables analogous to the ones used by J. Łukasiewicz's in his three-valued propositional logic. He declared that the systems were confluent but did not provide the proof. The adequacy of $\mathbf{K}_1$ with respect to a model based on boolean algebra, corresponding to Kalinowski's intuitions presented in an informal way in [4], has recently been proved in [9]. In the present paper we prove the adequacy of the two Kalinowski's original approaches.

The presented proof is not particularly difficult or innovative but we find it worth attention for several reasons. Firstly, the adequacy of the two presentations of $\mathbf{K}_1$ has not been proven yet. Secondly, this property is not obvious at first sight. In fact, it is somehow unexpected when one looks at the only axiom of the syntactic description of the system. The axiom does not indicate the trivalence of the logic since it resembles the excluded middle principle characteristic for the classical bivalent logic. Thirdly, the technique used in the proof, though known from the papers of S. Halldén [3] and L. Åquist [1], has been neglected and is worth revoking since it is an efficient and elegant formal tool. $\mathbf{K}_1$ logic is the simplest of the whole family of systems to which the technique can be applied (due to the limited language). Thus, the presented proof can be treated as a good introduction to the technique.

1. K_1 logic

1.1. Language

The language of the logic can be defined in Backus-Naur notation in the following way:

$$\varphi \ ::= \ \mathsf{P}(\alpha) \mid \neg\varphi \mid \varphi \to \varphi \tag{1.1}$$

$$\alpha \ ::= \ a_i \mid \overline{\alpha} \tag{1.2}$$

where a_i belongs to a set of basic action names $Act_0 = \{a_1, a_2, a_3, \dots\}$; $\mathsf{P}(\alpha)$ stands for 'α is permitted'; $\neg$ and $\to$ are operators of the classical propositional calculus (henceforward PC): negation and implication respectively.

For a fixed Act_0, the set of action names defined by (1.2) will be denoted by Act_K:

$$Act_K = Act_0 \cup \{\overline{a_1}, \overline{a_2}, \overline{a_3}, \dots, \overline{\overline{a_1}}, \overline{\overline{a_2}}, \overline{\overline{a_3}}, \dots \overline{\overline{\overline{a_1}}}, \overline{\overline{\overline{a_2}}}, \overline{\overline{\overline{a_3}}}, \dots\}$$

We shall call formulas of the form $\mathsf{P}(\alpha)$ atomic and use the symbol frm for the set of all formulas.

Other commonly used connectives of PC such as disjunction ($\vee$) and conjunction ($\wedge$) are defined in the standard way. Moreover, further deontic operators: obligation (O), forbiddance (F) and neutrality (N) will be used as defined notions.

1.2. Axiomatisation

Logic $\mathbf{K}_1$ is defined by the rule of Modus Ponens of the usual schema, the rule of double complement elimination of the following form:

$$\frac{\varphi(\overline{\overline{\alpha}})}{\varphi(\overline{\overline{\alpha}}//\alpha)}, \tag{1.3}$$

all PC theorems (more precisely all their substitutions in the language) as axioms and the only specific axiom:

$$\neg\mathsf{P}(\overline{\alpha}) \to \mathsf{P}(\alpha) \tag{1.4}$$

More precisely axiom (1.4) is an axiom schema in which one can put any action from Act_K in the place of α. Intuitively, the axiom states that either α is permitted or its negations is.

Obligation, forbiddance and neutrality are defined in the following way:

$$\mathsf{O}(\alpha) =_{df} \neg\mathsf{P}(\overline{\alpha}) \tag{1.5}$$

$$\mathsf{F}(\alpha) =_{df} \neg\mathsf{P}(\alpha) \tag{1.6}$$

$$\mathsf{N}(\alpha) =_{df} \mathsf{P}(\alpha) \land \mathsf{P}(\overline{\alpha}) \tag{1.7}$$

The following formulas follow straightforwardly from the axiom and the definitions:

$$\mathsf{O}(\alpha) \to \mathsf{P}(\alpha) \tag{1.8}$$

$$\mathsf{N}(\alpha) \to \mathsf{P}(\alpha) \tag{1.9}$$

$$\mathsf{O}(\alpha) \lor \mathsf{F}(\alpha) \lor \mathsf{N}(\alpha) \tag{1.10}$$

1.3. Matrix semantics

Kalinowski, inspired by Łukasiewicz, introduced a three-valued matrix system for his logic. The matrices are shown in Table 1.

The three values b, n and g, refer to bad, neutral and good actions respectively. (Kalinowski originally used Łukasiewicz's values 0, $\frac{1}{2}$ and 1 with the same interpretation.)

To make the adequacy consideration precise let us complement the matrices introduced by Kalinowski with a usual formal definition of a tautology of a matrix system appropriate for the matrices.

α	$\overline{\alpha}$
b	g
n	n
g	b

α	$\mathsf{P}(\alpha)$	$\mathsf{O}(\alpha)$	$\mathsf{F}(\alpha)$	$\mathsf{N}(\alpha)$
b	0	0	1	0
n	1	0	0	1
g	1	1	0	0

Table 1. Kalinowski's matrices for deontic logic

DEFINITION 1. *Deontic matrix M for $\mathbf{K}_1$ is a tuple*

$$M = \langle \mathcal{D}, \mathcal{R}, f^- \rangle,$$

where:

- $\mathcal{D} = \{b, n, g\}$ *is a set of deontic values;*
- $\mathcal{R} = \{P, O, F, N\}$ *is a set of functions from $\mathcal{D}$ to Fregean truth values $\{0, 1\}$, corresponding to the primitive and defined deontic operators;*
- $f^- : \mathcal{D} \longrightarrow \mathcal{D}$ *is a function attaching a deontic value to complex actions (the ones specified with the use of the complement operator).*

The content of the functions from the set $\mathcal{R}$ and of the function f^- is taken from the matrices in Table 1.

To define the notion of the tautology of the deontic matrix M we need to introduce a family of interpretation functions – $\mathcal{I} : Act_K \longrightarrow \mathcal{D}$ and valuation functions – v: $frm \longrightarrow \{0, 1\}$ based on them. The interpretation functions assign deontic values from $\mathcal{D}$ to basic actions from Act_0 and consequently to complex actions according to the function f^-. For each interpretation function $\mathcal{I}$ there exists a valuation function v, assigning truth values to atomic formulas according to functions from the set $\mathcal{R}$ and to complex formulas according to the usual truth tables of the classical propositional calculus.

A formula φ from frm is a tautology of M if and only if for each valuation function its value is 1.

2. Adequacy

The main concern of the present paper is to prove that the two presentations of $\mathbf{K}_1$ given by Kalinowski define the same logic. Formally we have to show that a formula φ is provable in the axiomatic version of $\mathbf{K}_1$ (is a theorem of $\mathbf{K}_1$) if and only if φ it is a tautology of the matrix system of $\mathbf{K}_1$.

To prove that every theorem is a tautology it is enough to show that (i) axioms and definitions are tautologies and (ii) the set of tautologies is closed under the rules of the logic. Checking (i) is a usual routine. The set of tautologies is closed under Modus Ponens since implication is classical and is closed under rule (1.3) since the matrix for the complement operator guarantees that for any action α, action $\overline{\overline{\alpha}}$ always has the same deontic value as α.

For the completeness part of the proof we use the method introduced by Halldén in [3]. To apply the method we have to be able to express the deontic values of actions in the language. The matrix defining the functions O, N and F corresponding to the operators O, N and F from Table 1 allows us to do that by connecting values g, n and b of an action α with the formulas $\mathsf{O}(\alpha)$, $\mathsf{N}(\alpha)$ and $\mathsf{F}(\alpha)$ respectively.

We can connect any formula φ and its interpretation $\mathcal{I}$ with a formula describing the interpretation for the action variables occurring in φ. Let us use the symbol $\varphi_{\mathcal{I}}$ for that formula.

Let $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ be all basic action names occurring in φ. Let further:

$$\varphi_i = \begin{cases} \mathsf{O}(a_i) & \text{if } \mathcal{I}(a_i) = g \\ \mathsf{N}(a_i) & \text{if } \mathcal{I}(a_i) = n \\ \mathsf{F}(a_i) & \text{if } \mathcal{I}(a_i) = b \end{cases}$$

for any i such that $1 \leq i \leq n$ and a fixed interpretation $\mathcal{I}$. Then, $\varphi_{\mathcal{I}}$ is the conjunction of all φ_i $(1 \leq i \leq n)$:

$$\varphi_{\mathcal{I}} = \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n$$

For example, let us consider

$$\varphi = (\mathsf{P}(a_1) \wedge \mathsf{P}(a_2)) \rightarrow (\neg\mathsf{O}(a_1) \vee \mathsf{P}(\overline{a_2}))$$

and the interpretation $\mathcal{I}$ such that $\mathcal{I}(a_1) = g$ and $\mathcal{I}(a_2) = b$. Then we have:

$$\varphi_{\mathcal{I}} = \mathsf{O}(a_1) \wedge \mathsf{F}(a_2)$$

The formula φ is used only to determine the set of variables that are used to construct the formula $\varphi_{\mathcal{I}}$.

We will now show that for any action name, α occurring in φ and any interpretation function $\mathcal{I}$ we have:

(A1) If $\mathcal{I}(\alpha) = g$, then $\varphi_{\mathcal{I}} \to \mathsf{O}(\alpha)$ is a theorem of $\mathbf{K}_1$.
(A2) If $\mathcal{I}(\alpha) = n$, then $\varphi_{\mathcal{I}} \to \mathsf{N}(\alpha)$ is a theorem of $\mathbf{K}_1$.
(A3) If $\mathcal{I}(\alpha) = b$, then $\varphi_{\mathcal{I}} \to \mathsf{F}(\alpha)$ is a theorem of $\mathbf{K}_1$.

Due to the definition of $\varphi_{\mathcal{I}}$, for any basic action name a_i occurring in φ, formulas from properties (A1), (A2) and (A3) are substitutions of PC tautologies. To extend the properties to arbitrary action names it is enough to notice that there exists a thesis of the system corresponding to each entry in the matrix for complement, i.e. the following formulas are provable:

$$\mathsf{F}(\alpha) \to \mathsf{O}(\overline{\alpha}) \tag{2.1}$$

$$\mathsf{N}(\alpha) \to \mathsf{N}(\overline{\alpha}) \tag{2.2}$$

$$\mathsf{O}(\alpha) \to \mathsf{F}(\overline{\alpha}) \tag{2.3}$$

To see that let us just replace operators F, O and N occurring in them with P, with the use of the definitions of the operators:

$$\neg\mathsf{P}(\alpha) \to \neg\mathsf{P}(\overline{\overline{\alpha}}) \tag{2.4}$$

$$\mathsf{P}(\alpha) \wedge \mathsf{P}(\overline{\alpha}) \to \mathsf{P}(\overline{\alpha}) \wedge \mathsf{P}(\overline{\overline{\alpha}}) \tag{2.5}$$

$$\neg\mathsf{P}(\overline{\alpha}) \to \neg\mathsf{P}(\overline{\alpha}) \tag{2.6}$$

Now, we can pass from conditions for action names (A1) - (A3) to the following conditions for formulas. For any φ_1 being a subformula of φ we have (v is a valuation based on interpretation $\mathcal{I}$):

(F1) If $v(\varphi_1) = 1$, then $\varphi_{\mathcal{I}} \to \varphi_1$ is a theorem of $\mathbf{K}_1$.
(F2) If $v(\varphi_1) = 0$, then $\varphi_{\mathcal{I}} \to \neg\varphi_1$ is a theorem of $\mathbf{K}_1$.

Let us first consider the atomic φ_1 of the form $\mathsf{P}(\alpha)$. In the case of (F1) $\mathcal{I}(\alpha)$ equals either g or n. Then $\varphi_{\mathcal{I}} \to \mathsf{O}(\alpha)$ or $\varphi_{\mathcal{I}} \to \mathsf{N}(\alpha)$ is provable in $\mathbf{K}_1$. Formula $\varphi_{\mathcal{I}} \to \mathsf{P}(\alpha)$ follows from both of them by (1.8) and (1.9) respectively. In the case of (F2) $\varphi_{\mathcal{I}} \to \mathsf{F}(\alpha)$ is provable in $\mathbf{K}_1$. $\varphi_{\mathcal{I}} \to \neg\mathsf{P}(\alpha)$ follows from it.

For complex formulas we can prove that both properties (F1) and (F2) hold by induction on the number of occurrences of negation and implication. It is enough to show that if properties (F1) and (F2) hold for subformulas of φ: φ_2 and φ_3, then they hold for $\neg\varphi_2$ and $\varphi_2 \to \varphi_3$. That fact is, however, straightforward in PC.

Thus, by (F1), if a formula φ is a tautology, then for any interpretation $\mathcal{I}$ we can prove $\varphi_{\mathcal{I}} \to \varphi$ in $\mathbf{K}_1$. To complete the adequacy proof we need to show that the disjunction of all $\varphi_{\mathcal{I}}$ for all interpretations is provable. That, however, follows from thesis (1.10). We just need to apply it to all basic action names occurring in φ, create a conjunction of such formulas and transform the resulting formula using the law of the distribution of conjunction with respect to disjunction.

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