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A REPRESENTATION FOR FINITE HILBERT ALGEBRAS

Abstract

In this work we show, that given a finite Hilbert algebra $\mathcal{A}$, there exists an anti-isomorphism of $\mathcal{A}$ in a subreduct of the algebra $\mathcal{D}(A)$ of the deductive systems of A, where $\mathcal{D}(A)$ is endowed with the implication defined between sets given by Horn.

1. Introduction and preliminaries

Hilbert algebras were introduced by Henkin and Skolem in the middle of the last century. It is well known that these algebras are the algebraic counterpart of the $\{\to\}$-fragment of the intuitionistic propositional calculus, which has been investigated by many authors. Diego proved, in [3], among other things, that they constitute a variety which will be denoted here by $\mathcal{H}$. Horn, in [6], proved that every Hilbert algebra can be immersed in an algebra of variety $\mathcal{H}$ whose underlying ordered structure is that of a lattice with last element. Marsden, in [7], defined the notion of compatible elements in a Hilbert algebra.

This article consists of three sections, the present one included. In Section 2, from a Hilbert algebra $\mathcal{A}$, we introduce the properties of the algebra $\mathcal{A}_{fin}$, whose supporting set is the set of all finite non-empty subsets of A, necessary to prove, among other results, those relative to the cardinality of certain sets. In the last section we establish that every finite Hilbert algebra, whose underlying ordered structure is that of a lattice, is anti-isomorphic to a subreduct of the algebra $\mathcal{D}(A) = \langle D(A), \succ\!\!\!\rightarrow, \cap, \sqcup, \{1\}, A\rangle$, in which $D(A)$ is the family of the deductive systems of A ordered by inclusion.

Let us recall that Hilbert algebras can be defined as algebras $\langle A, \to, 1\rangle$ of type $(2,0)$, in which the following conditions are verified:

(H1) $x \to (y \to x) = 1$,
(H2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
(H3) if $x \to y = 1$ and $y \to x = 1$, then $x = y$.

Some properties of Hilbert algebras are:

(H4) $x \to 1 = 1$,
(H5) $x \to x = 1$,
(H6) $1 \to x = x$,
(H7) the relation $\leq$ defined by $x \leq y$ if and only if $x \to y = 1$ is an order.

With respect to this order, 1 is the last element of A, and

(H8) if $x \leq y$, then $y \to z \leq x \to z$,
(H9) if $x \leq y$, then $z \to x \leq z \to y$.
(H10) $x \to (x \to y) = x \to y$,
(H11) $x \to (y \to z) = y \to (x \to z)$,
(H12) $x \to (y \to z) = (x \to y) \to (x \to z)$,
(H13) $((x \to y) \to y) \to y = x \to y$,
(H14) $x \leq (x \to y) \to y$.

Definition 1.1. *A subset D of a Hilbert algebra A is a deductive system of A if it satisfies:*

(1) $1 \in D$,
(2) $x \in D$ *and* $x \to y \in D$ *imply* $y \in D$.

If $T \subseteq A$, then $\bigcap\{D \in D(A) : T \subseteq D\}$ is called the deductive system generated by T and and it is denoted by $[T)$. If $T = \{t_1, \cdots, t_n\}$, then $[T) = \{x \in A : (t_1, \cdots, t_n; x) = 1\}$.

Next we will review some concepts relative to a Hilbert algebra $\mathcal{A}$ and include notions of the algebra $\mathcal{A}_{fin}$, indicated in [6] and adapted to our needs.

In [2] it is pointed out that $\langle A_{fin}, \cup, \{1\}\rangle$ is considered to be a lower semi-lattice with unit in which $\cup$ is the union of sets.

In A_{fin}, Horn defined two binary operations: 'addition' $(+)$ and 'implication' $(\succ)$; he also considered the relation $\approx$, defined as $L \approx K$ if and only if $K \succ L = \{1\}$ and $L \succ K = \{1\}$, for every $L, K \in A_{fin}$. Cīrulis mentioned that the relation $\approx$ is a congruence of the algebra $\mathcal{A}_{fin} = \langle A_{fin}, \succ, \cup, \{1\}\rangle$ of type $(2,2,0)$ and that there exists an immersion of $\mathcal{A}$ in $\mathcal{A}_{fin}/\approx$.

In section 2 we will review the concept of compatible elements and use it to prove that in each equivalence class of $\mathcal{A}_{fin}/\approx$ there is a unique set of minimal cardinality, which will turn out to be the irredundant generator of the deductive system that belongs to that equivalence class.

Now, in order to be able to describe the anti-isomorphism mentioned before we need to introduce new properties of the operations $\succ$ and $+$. We have adapted, according to our requirements, the definition of $\succ$ given in [6] as follows:

If $K = \{k_1, k_2, \ldots, k_m\}$ and L belong to A_{fin}, then

$$K \succ L := \{k_1 \to (\cdots \to (k_m \to l_j)\cdots) : l_j \in L\}.$$

Busneag represented, in a compact form, the implication between the elements of a Hilbert algebra in the following manner:

$$(t_1, t_2, \cdots, t_{n-1}; t_n) = \begin{cases} t_n, & \text{if} \quad n = 1, \\ t_1 \to (t_2, t_3, \cdots, t_{n-1}; t_n), & \text{if} \quad n > 1. \end{cases}$$

By means of this notation, the implication between sets may be described as follows:

$$K \succ L := \{(k_1, \cdots, k_m; l_j) : l_j \in L\}.$$

We will complete this section with the definition of the operation $+$, [6], and with properties of $+$ and of $\succ$.

If in a Hilbert algebra $\mathcal{A} = \langle A, \to, 1\rangle$ the underlying structure $\langle A, \leq\rangle$ is that of a join-semilattice with the order induced by $\to$, then it is possible to define, in accordance with Horn, the operation $+$ by:

$$K + L = \{k \vee l : k \in K,\ l \in L\},$$

where $k \vee l$ is the supremum of k and l in $\langle A, \leq\rangle$.

Some of the properties of $\succ$ and of $+$, [6], are:

(Ho 1) $K \succ (L \succ M) = (K \cup L) \succ M = L \succ (K \succ M)$,
(Ho 2) $K \succ (L \cup M) = (K \succ L) \cup (K \succ M)$,
(Ho 3) $K \succ \{1\} = \{1\}$, and $\{1\} \succ K = K$,
(Ho 4) if $K \subset L$, then $L \succ K = \{1\}$,
(Ho 5) the relation defined by $K \leq L$ if and only if $K \succ L = \{1\}$ is reflexive and transitive,
(Ho 6) if $M \leq K$ and $M \leq L$, then $M \leq K \cup L$,
(Ho 7) if $K \leq L$, then $K \cup M \leq L \cup M$,
(Ho 8) $L \succ (K \succ L) = \{1\}$,
(Ho 9) $K \succ \{t\} \leq (K + L) \succ (\{t\} + L)$, with $t \in A$,
(Ho 10) $K \succ (K + L) = \{1\}$,
(Ho 11) $((K \succ M) \succ (L \succ M)) \succ ((K + L) \succ M) = \{1\}$.

2. The algebra $\mathcal{A}_{fin}/\approx$

Lemma 2.1 is used to prove that the relation $\approx$, defined in section 1, is a congruence of the algebra $\mathcal{A}_{fin} = \langle A_{fin}, \succ, \cup, +, \{1\}\rangle$.

LEMMA 2.1. *If $K, L, M \in A_{fin}$, then*

(Ho 12) $L \subset K$ *implies* $L \succ M \leq K \succ M$,
(Ho 13) $(K \succ L) \succ (K \succ M) = K \succ (L \succ M)$,
(Ho 14) $L \subset K$ *implies* $M \succ K \leq M \succ L$,
(Ho 15) $K \succ L \leq (K + M) \succ (L + M)$,
(Ho 16) $(K + K) \succ K = \{1\}$.

PROOF:

(Ho 12)
(1) $L \subset K$, [hyp.]
(2) there is $T \in A_{fin}$ such that $K = L \cup T$, [(1)]
(3) $K \succ M = (L \cup T) \succ M$ [(2)]
$\quad = T \succ (L \succ M)$, [(Ho 1)]
(4) $(L \succ M) \succ (T \succ (L \succ M)) = \{1\}$, [(Ho 8)]
(5) $L \succ M \leq T \succ (L \succ M)$, [(4), (Ho 5)]
(6) $L \succ M \leq K \succ M$. [(5), (3)]

(Ho 13) Let $K, L, M \in A_{fin}$ with $K = \{k_1, \cdots, k_p\}$ and $L = \{l_1, \cdots, l_q\}$.

$(K \succ L) \succ (K \succ M) = ((K \succ L) \cup K) \succ M$ [(Ho 1)]
$= \{(k_1, \cdots, k_p; l_1), \cdots, (k_1, \cdots, k_p; l_q), k_1, \cdots, k_p\} \succ M$
$= \{\big((k_1, \cdots, k_p; l_1), \cdots, (k_1, \cdots, k_p; l_q), k_1, \cdots, k_p; m_i\big) : m_i \in M\}$
$= \{\big((k_1, \cdots, k_p; l_1), \cdots, (k_1, \cdots, k_p; l_q); (k_1, \cdots, k_p; m_i)\big) : m_i \in M\}$
$= \{(k_1, \cdots, k_p, l_1 \cdots, l_q; m_i) : m_i \in M\}$ [(H12)]
$= (K \cup L) \succ M$
$= K \succ (L \succ M)$. [(Ho 1)]

(Ho 14) (1) $L \subset K$, [hyp.]
(2) $K \succ L = \{1\}$, [(1)]
(3) $(M \succ K) \succ (M \succ L) = M \succ (K \succ L)$, [(Ho 13)]
(4) $M \succ \{1\} = (M \succ K) \succ (M \succ L)$, [(2), (3)]
(5) $\{1\} = (M \succ K) \succ (M \succ L)$, [(4), (Ho 3)]
(6) $M \succ K \leq M \succ L$. [(5)]

(Ho 15) The proof is by induction over the cardinal of L.
(1) $K \succ \{x\} \leq (K + M) \succ (\{x\} + M)$, whichever $x \in A$, [(Ho 9)]
(2) $K \succ P \leq (K+M) \succ (P+M)$, with $P \in A_{fin}$ such that $|P| > 1$, [ind. hyp.]
(3) $L = P \cup \{x\}$, [hyp.]
(4) $K \succ L = K \succ (P \cup \{x\})$ [(3)]
$= (K \succ P) \cup (K \succ \{x\})$ [(Ho 2)]
$\leq ((K + M) \succ (P + M)) \cup ((K + M) \succ (\{x\} + M))$ [(2), (1)]
$= (K + M) \succ ((P + M) \cup (\{x\} + M))$, [(Ho 2)]
(5) $(P + M) \cup (\{x\} + M) =$
$= \{p_j \vee m_k : p_j \in P, m_k \in M\} \cup \{x \vee m_t : m_t \in M\}$
$= \{q_t \vee m_s : q_t \in P \text{ or } q_t = x, m_s \in M\}$
$= L + M$,
(6) $K \succ L \leq (K + M) \succ (L + M)$. [(4), (5)]

(Ho 16) (1) $k \in K$, [hyp.]
(2) $k = k \vee k$,
(3) $k \in K + K$, [(2)]

(4) $K \subseteq K + K$, [(1), (3)]
(5) $(K + K) \succ K = 1$. [(Ho 4)]

□

By (Ho 10) and (Ho 11), $K + L$ is the least upper bound of K and L. $K + L$ is not a supremum.

Lemma 2.2. *The relation $\approx$, defined by $K \approx L$ if and only if $K \succ L = \{1\}$ and $L \succ K = \{1\}$, is a congruence on $\mathcal{A}_{fin}$.*

Proof: It is routine to prove that $\approx$ is an equivalence relation. We will prove that it is compatible with the operations $\succ$, $\cup$ and $+$.

(a) If $K \approx L$ and $P \approx Q$, then $K \succ P \approx L \succ Q$. Indeed,

(1) $K \succ L = \{1\}$, $L \succ K = \{1\}$, [hyp.]
(2) $P \succ Q = \{1\}$, $Q \succ P = \{1\}$. [hyp.]
(3)
$$\begin{aligned}(K \succ P) \succ (L \succ Q) &= L \succ ((K \succ P) \succ Q) && [(\text{Ho } 1)]\\
&= (L \succ (K \succ P)) \succ (L \succ Q) && [(\text{Ho } 13)]\\
&= ((L \succ K) \succ (L \succ P)) \succ (L \succ Q) && [(\text{Ho } 13)]\\
&= (\{1\} \succ (L \succ P)) \succ (L \succ Q) && [(1)]\\
&= (L \succ P) \succ (L \succ Q) && [(\text{Ho } 3)]\\
&= L \succ (P \succ Q) && [(\text{Ho } 1)]\\
&= L \succ \{1\} && [(2)]\\
&= \{1\}. && [(\text{Ho } 3)]\end{aligned}$$

In a similar way we prove that:

(4) $(L \succ Q) \succ (K \succ P) = \{1\}$,
(5) $K \succ P \approx L \succ Q$. [(3), (4)]

(b) If $K \approx L$ and $P \approx Q$, then $K \cup P \approx L \cup Q$. Indeed,

(1)
$$\begin{aligned}(K \cup P) \succ (L \cup Q) &= K \succ (P \succ (L \cup Q)) && [(\text{Ho } 1)]\\
&= (K \succ P) \succ (K \succ (L \cup Q)) && [(\text{Ho } 13)]\\
&= (K \succ P) \succ ((K \succ L) \cup (K \succ Q)) && [(\text{Ho } 2)]\\
&= (K \succ P) \succ (\{1\} \cup (K \succ Q))\\
&= ((K \succ P) \succ \{1\}) \cup ((K \succ P) \succ (K \succ Q)) && [(\text{Ho } 2)]\\
&= \{1\} \cup (K \succ (P \succ Q)) && [(\text{Ho } 3), (\text{Ho } 13)]\\
&= \{1\} \cup (K \succ \{1\}) && [\text{hyp.}]\end{aligned}$$

$= \{1\} \cup \{1\}$ [(Ho 3)]
$= \{1\}$.

By means of the same procedure we prove that:

(2) $(L \cup Q) \succ (K \cup P) = \{1\}$.
(3) $K \cup Q \approx L \cup P$. [(1), (2)]

(c) Finally, we prove that $\approx$ is compatible with $+$.

(1) $K \approx S$, [hyp.]
(2) $K \succ S = \{1\}$, [(1)]
(3) $K \succ S \leq (K + L) \succ (S + L)$, [(Ho 15)]
(4) $\{1\} = (K + L) \succ (S + L)$. [(2), (3)]

In an analogous way we obtain:

(5) $(S + L) \succ (K + L) = \{1\}$.
(6) $S + L \approx K + L$. [(4), (5)]

□

Obviously, $\langle A_{fin}/ \approx, \succ, [\{1\}]_{\approx} \rangle$ is a Hilbert algebra. The order in $A_{fin}/\approx$ is that induced by the preorder defined in (Ho 5).

In [2], there are five equivalent conditions relative to compatible elements. In this paper we select the following as definition:

Definition 2.3. *Let a and b be elements of a Hilbert algebra. If there exists the meet $a \wedge b$, of a and b, then a and b are said to be compatible if and only if $a \to (a \wedge b) = a \to b$.*

If a and b are compatible, we will write $a\,Cb$.

Lemma 2.4. [7] *If a and b are elements of a Hilbert algebra A and $a \wedge b \in A$, then the following conditions are equivalent:*

(i) $a\,Cb$,
(ii) $(a \wedge b) \to x = a \to (b \to x)$, *for every $x \in A$.*

Lemma 2.5 and Lemma 2.9 show properties inherent to the cardinality of the sets that belong to the same class of equivalence of $A_{fin}/\approx$, and they are employed to characterize the equivalence classes by means of deductive systems.

In what follows we will denote by $\mathcal{HO}$ the class of finite Hilbert algebras with a lattice as underlying structure. Besides, we will denote the equivalence class of K by $\overline{K}$ and the cardinality of K, by $|K|$.

We will use the following facts:

LEMMA 2.5. *If $\overline{K} \in A_{fin}/\approx$ and $T = \{t_1, \cdots, t_n\}$ is a set of minimal cardinality in $\overline{K}$, then t_i is not compatible with t_j, $i \neq j$.*

LEMMA 2.6. [6] *If $\mathcal{A} \in \mathcal{HO}$, then $x \wedge (x \to y) \leq y$ for all $x, y \in A$.*

LEMMA 2.7. *If $\mathcal{A} \in \mathcal{HO}$ and $T \in A_{fin}$, then T and $[T)$ are in the same equivalence class.*

PROOF:

(1) $T = \{t_1, t_2, \cdots, t_m\} \in A_{fin}$, [hyp.]
(2) $[T) \succ T = \{1\}$, [(Ho 4)]
(3) $T \succ [T) = \{(t_1, t_2, \cdots, t_m; x) : x \in [T)\}$
$\quad = \{1\}$,
(4) $T \approx [T)$, [(2), (3)]
(5) T and $[T)$ belong to the same class. [(4)]

□

LEMMA 2.8. *In each equivalence class of $A_{fin}/\approx$ there exists one and only one deductive system.*

The proof is immediate.

Even though all the sets that belong to the same class generate the same deductive system, they may have different cardinality, but the one of minimal cardinality is unique, which is demonstrated in the following lemma.

LEMMA 2.9. *If $\mathcal{A} \in \mathcal{HO}$, then in each equivalence class of $A_{fin}/\approx$ there is a unique element of minimal cardinality.*

Proof: Let $\overline{K} \in A_{fin}/\!\approx$.

(1) If $\{a\}$ and $\{b\} \in \overline{K}$, then $a = b$.

(2) If $\{a_1, a_2, \cdots, a_n\}$, $\{b_1, b_2, \cdots, b_n\}$ are two sets of minimal cardinality in $\overline{K}$, then:

(2.1) $\{a_1, a_2, \cdots, a_n\} \approx \{b_1, b_2, \cdots, b_n\}$,

(2.2) $\{a_1, a_2, \cdots, a_n\} \succ \{b_1, b_2, \cdots, b_n\} = \{1\}$, [(2.1)]

(2.3) $(a_1, a_2, a_3, \cdots, a_n; b_t) = 1,\ \ t \in \{1, 2, \cdots, n\}$, [(2.2)]

(2.4) $a_1 \leq (a_2, a_3, \cdots, a_n; b_t),\ \ t \in \{1, 2, \cdots, n\}$, [(2.3)]

(2.5) $a_2 \wedge a_1 \leq a_2 \wedge (a_2, a_3, \cdots, a_n; b_t),\ \ t \in \{1, 2, \cdots, n\}$ [(2.4)]
$\leq (a_3, \cdots, a_n; b_t),\ \ t \in \{1, 2, \cdots, n\}$, [(2.4), Lemma 2.6]

(2.6) $(a_1 \wedge a_2) \to (a_3, \cdots, a_n; b_t) = 1,\ \ t \in \{1, 2, \cdots, n\}$, [(2.5)]

(2.7) $a_1 \to (a_2 \to (a_3, \cdots, a_n; b_t)) = (a_1 \wedge a_2) \to (a_3, \cdots, a_n; b_t)$,
$t \in \{1, 2, \cdots, n\} = 1$, [(2.3), (2.6)]

(2.8) $a_2\, C a_1$, [(2.7), Lemma 2.4]

which contradicts Lemma 2.5.

□

3. Anti-isomorphism between the algebras $\mathcal{D}(A)$ and $\mathcal{C}(A_{fin})$

The aim of this section is to prove that if $\mathcal{A} \in \mathcal{HO}$, then $\mathcal{D}(A) = \langle D(A), \not\succ, \cap, \sqcup, \{1\}, A\rangle$ and $\mathcal{C}(A_{fin}) = \langle A_{fin}/\!\approx, \succ, +, \cup, \overline{\{1\}}, \overline{\{0\}}\,\rangle$, both algebras of type $(2, 2, 2, 0, 0)$, are anti-isomorphic, with $D_1 \not\succ D_2 := [D_1 \succ D_2)$, and $D_1 \sqcup D_2 := [D_1 \cup D_2)$.

The result that follows is used to prove the main theorem in this work.

Lemma 3.1. *If D_1 and $D_2 \in D(A)$, then $D_1 \cap D_2 = D_1 + D_2$.*

Proof:

(1) $D_1, D_2 \in \mathcal{D}(A)$, [hyp.]

(2) $x \in D_1 \cap D_2$, [hyp.]

(3) $x \in D_1$ and $x \in D_2$, [(2)]

(4) $x = x \vee x$,

(5) $x \in D_1 + D_2$, [(4), (3)]

(6) $D_1 \cap D_2 \subseteq D_1 + D_2$, [(2), (5)]

(7) $y \in D_1 + D_2$, [hyp.]
(8) $y = z \vee t$ with $z \in D_1$ and $t \in D_2$, [(7)]
(9) $z \leq y$, $t \leq y$, [(8)]
(10) $y \in D_1$, $y \in D_2$, [(9), (1)]
(11) $y \in D_1 \cap D_2$, [(10)]
(12) $D_1 + D_2 \subseteq D_1 \cap D_2$, [(7), (11)]
(13) $D_1 \cap D_2 = D_1 + D_2$. [(6), (12)]

□

THEOREM 3.2. *Let $\mathcal{A} \in \mathcal{HO}$. Then, the application $\varphi\colon \mathcal{D}(A) \longrightarrow \mathcal{C}(A_{fin})$, defined by $\varphi(D) = \overline{D}$, is an isomorphism such that $D_1 \subseteq D_2$ if and only if $\varphi(D_2) \leq \varphi(D_1)$.*

PROOF:

Let $\mathcal{A} \in \mathcal{HO}$, φ: $\mathcal{D}(A) \longrightarrow \mathcal{C}(A_{fin})$ defined by $\varphi(D) = \overline{D}$, and D_1 and $D_2 \in D(A)$.

(1) The injectivity of φ is a consequence of the Lemma 2.7.

(2) φ is surjective. Indeed,

(2.1) $\overline{K} \in A_{fin}/\approx$, [hyp.]
(2.2) there exists $L \in \overline{K}$ such that L is of minimal cardinality, [Lemma 2.9]
(2.3) $L \approx [L)$, [Lemma 2.7]
(2.4) $\overline{[L)} \in \overline{K}$, [(2.3), (2.2)]
(2.5) $\overline{[L)} = \overline{K}$, [(2.4)]
(2.6) $\overline{K} = \varphi([L))$. [(2.5)]

(3) Next we prove that φ is a homomorphism.

(3.1) $\varphi(D_1 \mathrel{\succ\!\!\!\succ} D_2) = \varphi([D_1 \succ D_2))$
$= \overline{[D_1 \succ D_2)}$ [def.]
$= \overline{D_1 \succ D_2}$ [Lemma 2.7]
$= \overline{D_1} \succ \overline{D_2}$
$= \varphi(D_1) \succ \varphi(D_2)$. [def.]

(3.2) $\varphi(D_1 \cap D_2) = \varphi(D_1 + D_2)$ [Lemma 3.1]
$= \overline{(D_1 + D_2)}$
$= \overline{D_1} + \overline{D_2}$
$= \varphi(D_1) + \varphi(D_2)$.

(3.3) $\varphi(D_1 \sqcup D_2) = \overline{D_1 \sqcup D_2}$

$$
\begin{aligned}
&= \overline{D_1 \cup D_2} && \text{[Lemma 2.7]}\\
&= \overline{D_1} \cup \overline{D_2} && \text{[Lemma 2.2]}\\
&= \varphi(D_1) \cup \varphi(D_2).
\end{aligned}
$$

(3.4) $\varphi(\{1\}) = \overline{\{1\}}$, by the definition of φ.

(3.5) $\varphi(A) = \overline{A}$ [def.]

$= \overline{\{0\}}$. [$A \approx \{0\}$]

(4) In what follows we will prove that φ reverses the order.

(4.1) (i) $D_1 \subseteq D_2$, [hyp.]

(ii) $D_2 \succ D_1 = \{1\}$, [(i), (Ho 4)]

(iii) $\overline{D_2 \succ D_1} = \overline{\{1\}}$, [(ii)]

(iv) $\overline{D_2} \succ \overline{D_1} = \overline{\{1\}}$, [(iii), Lemma 2.9]

(v) $\overline{D_2} \leq \overline{D_1}$. [(iii), (iv)]

(4.2) (i) $\overline{D_2} \leq \overline{D_1}$, [hyp.]

(ii) $\overline{D_2} = \overline{D_2} \cup \overline{D_1}$ [(i)]

$= \overline{D_2 \cup D_1}$,

(iii) $D_2 \approx D_2 \cup D_1$ [(ii)]

$\approx D_2 \sqcup D_1$,

(iv) $D_2 = D_2 \sqcup D_1$, [(iii), Lemma 2.8]

(v) $D_1 \subseteq D_2$.

□

As every $\mathcal{A} \in \mathcal{HO}$ is isomorphic to a subreduct of the $\mathcal{C}(A_{fin})$ ([6]) and this algebra is anti-isomorphic to $D(A)$, we have that

Corollary 3.3. *Every $\mathcal{A} \in \mathcal{HO}$ is anti-isomorphic to a subreduct of the algebra $\mathcal{D}(A)$.*

Remark 3.4. *If we consider the dual lattice of the reduct $\langle D(A), \cap, \sqcup, \{1\}, A\rangle$ of the algebra $\mathcal{D}(A)$ and define $D_i \not\succ D_j = \max\{D' : D_i \sqcup D' \leq D_j\}$, we obtain a Heyting algebra in which $D_i \leq D_j$ if and only if $D_i = D_i \sqcup D_j$.*

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