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ABOUT SIMULATING POLYADIC FRAMES

Abstract

This paper is devoted to the problem of expressivity of polyadic frame by the frame containing unary modalities only. Our approach differs from [2] because our class of simulation frames is modally definable.

Keywords: modal logic, polyadic logic, Kripke models, frames.

1. Introduction

Modal logic is one of the most important areas for all of formal logic. It has very relevant consequences for philosophical logic. Philosophy gives as good as it gets and provides a great inspiration for its formal counterpart. The advantages of investigations concerning modal logic are more than purely theoretical. Modal logic has many applications in linguistics, computer science, artificial intelligence and others. However, the abovementioned branches of science often require some non-standard modal languages. There are many examples of extensions of unary modal logic. One of such extensions are logics expressed in languages containing modal operators with arities greater than 1. A question could be asked – how important is such an enlarging of languages? Naturally this question does not touch such modal logics as SINCE/UNTIL at all because they involve a nonstandard definition of the satisfiability of a formula at a point/world.

Paper [2] contains a description of the way of simulating polyadic modal logic, by modal logic that does not contain a modality of arity greater than 1. In the construction a mapping $^{\bullet}$ has been used, which takes polyadic frame $\mathfrak{F}$ of a fixed type as an argument, and associates with it another frame $\mathfrak{F}^\bullet$ that contains binary relations only. Moreover, the following condition is valid:

$$\mathfrak{F} \Vdash \psi \text{ iff } \mathfrak{F}^\bullet \Vdash \psi^\Diamond \tag{1.1}$$

where $^\Diamond$ is some defined earlier translation of the modal language for the polyadic frame $\mathfrak{F}$ into the modal language of the frame $\mathfrak{F}^\bullet$. So, it can be said that frame $\mathfrak{F}^\bullet$ simulates $\mathfrak{F}$. The class of *simulation frames* is defined in [2] as the class of isomorphic copies of all frames that simulate some polyadic frame:

let τ be some modality type (see [1]), then

$$\mathfrak{G} \text{ is a } \textit{simulation frame} \text{ iff there exists } \mathfrak{F} \text{ of type } \tau \text{ such that } \mathfrak{G} \cong \mathfrak{F}^\bullet \tag{1.2}$$

Unfortunately, a class of simulating frames is not modally definable. In this paper we present another way to simulate the frames, such that the class of simulating frames and their isomorphic copies can be characterized by a finite set of modal formulas. Moreover, the present construction has slightly reduced a number of modalities and a number of worlds occurring in a simulating frame. For example, in [2] for a given frame $\mathfrak{F} = (W, r_R)_{R\in O}$ $\mathfrak{F}^\bullet$ contains $\sum\limits_{R\in O}(1+\rho(R))$ binary relations (where $\rho(R)$ is an arity of the modality R). The construction presented in the next chapter (or rather its easy generalization) produces $4 + 2\cdot|O|$ modal operators (here $|O|$ stands for a number of modalities). Thus it does not depend on the arity of a particular modality.

2. Modal logic

For our aims we will use a modified version of a notation contained in [1]. We think this notation is the most proper for the multimodal case.
Let O be non-empty set of modalities and $\rho : O \to \mathbb{N}$. By a *modal similarity type* we understand a tuple $\tau = (O, \rho)$. Define a modal language $ML(\tau, \Phi)$ as a smallest set X containing propositional variables Φ and fulfilling

$$\phi,\ \psi \in X \text{ implies } \neg\phi,\ \phi\wedge\psi,\ \phi\vee\psi,\ \phi\to\psi \in X \tag{2.1}$$

and for $R \in O$

$$\phi_1, \ldots, \phi_{\rho(R)} \in X \text{ implies } \langle R\rangle(\phi_1, \ldots, \phi_{\rho(R)}),\ [R](\phi_1, \ldots, \phi_{\rho(R)}) \in X. \tag{2.2}$$

By *a frame for* τ (or τ*-frame*) we shall understand a tuple $\mathfrak{F} = (W, r_R)_{R\in O}$, where W is non-empty set, and for every $R \in O$ $r_R \subseteq W^{\rho(R)+1}$.

By *Kripke model for* τ (or *Kripke* τ*-model*) we shall understand a tuple $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is the frame for τ and $V : \Phi \to \mathcal{P}(W)$. In the sequel we will identify a modality symbol and a corresponding relational symbol, and the frame will be presented in the form $\mathfrak{F} = (W, R_1, \ldots, R_k)$ instead of $\mathfrak{F} = (W, r_{R_1}, \ldots, r_{R_k})$.
Define a relation $\Vdash$:

$$\begin{array}{rll} \mathfrak{M}, w \Vdash p & \text{iff} & p \in V(p) \\ \mathfrak{M}, w \Vdash \neg\phi & \text{iff} & \mathfrak{M}, w \not\Vdash \phi \\ \mathfrak{M}, w \Vdash \phi \wedge \psi & \text{iff} & \mathfrak{M}, w \Vdash \phi \ \&\ \mathfrak{M}, w \Vdash \psi \\ \mathfrak{M}, w \Vdash \phi \vee \psi & \text{iff} & \mathfrak{M}, w \Vdash \phi \ \text{ or } \ \mathfrak{M}, w \Vdash \psi \\ \mathfrak{M}, w \Vdash \phi \to \psi & \text{iff} & \mathfrak{M}, w \not\Vdash \phi \ \text{ or } \ \mathfrak{M}, w \Vdash \psi \end{array} \tag{2.3}$$

and for $R \in O$

$$\begin{array}{rll} \mathfrak{M}, w \Vdash \langle R\rangle(\phi_1, \ldots, \phi_{\rho(R)}) & \text{iff} & \text{there exist } v_1, \ldots, v_{\rho(R)} \in W \\ \text{such that } Rwv_1 \ldots w_{\rho(R)} & \text{and} & \forall_{i\in\{1,\ldots,\rho(R)\}} \mathfrak{M}, v_i \Vdash \phi_i \end{array} \tag{2.4}$$

$$\begin{array}{rll} \mathfrak{M}, w \Vdash [R](\phi_1, \ldots, \phi_{\rho(R)}) & \text{iff} & \text{for every } v_1, \ldots, v_{\rho(R)} \in W, \\ Rwv_1 \ldots w_{\rho(R)} & \text{implies} & \exists_{i\in\{1,\ldots,\rho(R)\}} \mathfrak{M}, v_i \Vdash \phi_i \end{array} \tag{2.5}$$

3. The simulating of polyadic frames

At first put $W^* = \bigcup\{W^k : k \in \mathbb{N}\}$. By convention we take $W^0 = \{\langle\rangle\}$, where $\langle\rangle$ is the tuple of lenght 0. If $\overline{w} = \langle w_1, \ldots, w_n\rangle$ and $\overline{v} = \langle v_1, \ldots, v_k\rangle$ are elements of W^*, then $\overline{wv}$ will stand for $\langle w_1, \ldots, w_n, v_1, \ldots, v_k\rangle$.

Let $\tau = \langle\{R\}, \rho\rangle$, where $\rho(R) = n > 1$. Put $\overline{\tau} = \langle\{\overline{R}, \overline{R}^{\sim}, H, H^{\sim}, T, T^{\sim}\}, \overline{\rho}\rangle$ where: $\overline{\rho}(\overline{R}) = \overline{\rho}(\overline{R}^{\sim}) = \overline{\rho}(H) = \overline{\rho}(H^{\sim}) = \overline{\rho}(T) = \overline{\rho}(T^{\sim}) = 1$.

For Kripke τ-model $\mathfrak{M} = (W, R, V)$, where $R \subseteq W^{n+1}$ put:

$\overline{W} = \{\langle w_1, w_2, \ldots, w_k\rangle \in W^* : 1 < k \leq n,\ \exists_{w_0, w_{k+1}, \ldots, w_n \in W} Rw_0w_1 \ldots w_n\}$
$\cup \{\langle w\rangle : w \in W\}$
$\overline{R} = \{\langle\langle w_0, v_1, \ldots, v_k\rangle, \langle w_1, \ldots, w_n\rangle\rangle \in \overline{W} \times \overline{W} :\ k \geq 0,\ \ Rw_0w_1 \ldots w_n\}.$
$H = \{\langle w_0, \ldots, w_k, w_{k+1}\rangle, \langle w_{k+1}\rangle : k > 0,\ \langle w_0, \ldots, w_k, w_{k+1}\rangle \in \overline{W}\}$
$T = \{\langle w_0, \ldots, w_k, w_{k+1}\rangle, \langle w_0, \ldots, w_k\rangle : k > 0,\ \langle w_0, \ldots, w_k, w_{k+1}\rangle \in \overline{W}\}$

$$\overline{V}(p) = [\bigcup_{0 \leq k < n} (V(p) \times W^k)] \cap \overline{W}.$$

Let $\overline{\tau} = (\{\overline{R}, \overline{R}^{\sim}, H, H^{\sim}, T, T^{\sim}\}, \overline{\rho})$, where $\overline{\rho}$ constantly equals 1. Define Kripke model for $\overline{\tau}$: $\overline{\mathfrak{M}} = (\overline{W}, \overline{R}, \overline{R}^{\sim}, H, H^{\sim}, T, T^{\sim}, \overline{V})$ where $\sim$ stands for convers of relation, i.e. $J^{\sim}xy$ iff Jyx for any binary relation J.

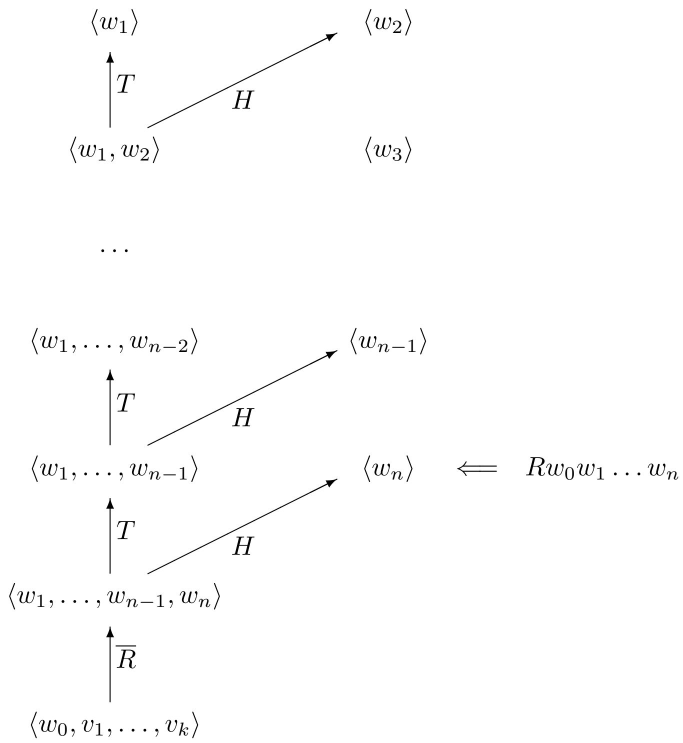
Put

$$\begin{array}{rcl} \eta(p) & := & p \\ \eta(\neg\phi) & := & \neg\eta(\phi) \\ \eta(\phi \wedge \psi) & := & \eta(\phi) \wedge \eta(\psi) \\ \eta(\langle R\rangle(\psi_1, \ldots, \psi_n)) & := & \langle\overline{R}\rangle[D^{n-1}(\eta(\psi_1), \ldots, \eta(\psi_n))] \end{array} \tag{3.1}$$

where $D^0(\phi_0) = \phi_0,\ D^k(\phi_0, \phi_1, \ldots, \phi_k) = \langle T\rangle[D^{k-1}(\phi_0, \phi_1, \ldots, \phi_{k-1}))] \wedge \langle H\rangle(\phi_k)$.
According to the laws of classical logic η can be extended on the rest of connectives:

$$\begin{array}{rcl} \eta(\phi \vee \psi) & := & \eta(\neg(\neg\phi \wedge \neg\psi)) \\ \eta(\phi \to \psi) & := & \eta(\neg\phi \vee \psi) \\ \eta([R](\psi_1, \ldots, \psi_n)) & := & \eta(\neg\langle R\rangle(\neg\psi_1, \ldots, \neg\psi_n)) \end{array} \tag{3.2}$$

Before we present the formal proofs we shall try to explain the construction.
The set $\overline{W}$ consists of the elements of the form $\langle w_0, w_1, \ldots, w_k\rangle$ where $w_0, w_1, \ldots,$
$w_k \in W$ for some $k \leq 0$. Every fact $Rw_0w_1 \ldots w_n$ in $\mathfrak{M}$ has its counterpart of the form $\overline{R}\langle w_0v_1 \ldots v_k\rangle\langle w_1w_2 \ldots w_n\rangle$ in $\overline{\mathfrak{M}}$. Moreover, we need some means to see if $\overline{Rwv}$, then $\overline{v}$ has the correct length: i.e. n and indeed $Rw_0v_1 \ldots v_n$ (where $\overline{v} = \langle v_1, v_2, \ldots, v_n\rangle$). For this aim, we have introduced two additional relations T (tail) and H (head). These relations allow us to decompose $\overline{v}$ into one element sequences, which is explained by the following diagram:



THEOREM 3.1. *$\mathfrak{M}, w \Vdash \phi$ iff $\overline{\mathfrak{M}}, \langle w\rangle\overline{u} \Vdash \eta(\phi)$ for every $\langle w\rangle\overline{u} \in \overline{W}$.*

PROOF: By induction on the length of formula ϕ. If $\phi \in Var$ then thesis follows from the definition of $\overline{V}$. The case of boolean connectives is immediate. So, let $\mathfrak{M}, w \Vdash \langle R\rangle(\psi_1, \ldots, \psi_n)$. Thus there are $v_1, \ldots, v_n \in W$ such that $Rwv_1 \ldots v_n$ and $\mathfrak{M}, v_i \Vdash \psi_i$ for $i = 1, \ldots, n$. We can note that $\langle v_1, \ldots v_i\rangle \in \overline{W}$ and $\overline{\mathfrak{M}}, \langle v_i\rangle \Vdash \eta(\psi_i)$ for $i = 1, \ldots, n$. We are showing that $\overline{\mathfrak{M}}, \langle v_1, \ldots, v_i\rangle \Vdash D^{i-1}(\eta(\psi_1), \ldots, \eta(\psi_i))$. At this stage we know that $\overline{\mathfrak{M}}, \langle v_1\rangle \Vdash D^0(\eta(\psi_1))$.

For the inductive step: by definition we have $T\langle v_1, \ldots, v_i, v_{i+1}\rangle\langle v_1, \ldots, v_i\rangle$ and $\overline{\mathfrak{M}}, \langle v_1, \ldots, v_i\rangle \Vdash D^{i-1}(\eta(\psi_1), \ldots, \eta(\psi_i))$ by the induction hypothesis. However, this implies $\overline{\mathfrak{M}}, \langle v_1, \ldots, v_i, v_{i+1}\rangle \Vdash \langle T\rangle D^{i-1}(\eta(\psi_1), \ldots, \eta(\psi_i))$.

Moreover,
$H\langle v_1,\ldots,v_i,v_{i+1}\rangle\langle v_{i+1}\rangle$ and $\overline{\mathfrak{M}},\langle v_{i+1}\rangle \Vdash \eta(\psi_{i+1})$. Summarizing $\overline{\mathfrak{M}},\langle v_1,\ldots,v_i,v_{i+1}\rangle \Vdash \langle T\rangle D^{i-1}(\eta(\psi_1),\ldots,\eta(\psi_i)) \wedge \langle H\rangle(\eta(\psi_{i+1}))$ and we are done, because $\overline{R}(\langle w\rangle\overline{u})\langle v_1,\ldots,v_n\rangle$.

For the other direction:

Let $\overline{\mathfrak{M}},\langle w\rangle\overline{u} \Vdash \langle\overline{R}\rangle[\langle T\rangle D^{n-1}(\eta(\psi_1),\ldots,\eta(\psi_{n-1}))\wedge\langle H\rangle(\psi_n)]$, which implies that $\overline{R}(\langle w\rangle\overline{u})\langle v_1\ldots v_n\rangle$ and $\overline{\mathfrak{M}},\langle v_1,\ldots,v_n\rangle \Vdash \langle T\rangle D^{n-1}(\eta(\psi_1),\ldots,\eta(\psi_{n-1}))\wedge\langle H\rangle(\psi_n)$ for some $v_1,\ldots,v_n$. However, the definition of $\overline{R}$ implies that $Rwv_1\ldots v_n$. We are going to show that

$$\overline{\mathfrak{M}},\langle v_1,\ldots,v_i\rangle \Vdash \langle T\rangle D^{i-1}(\eta(\psi_1),\ldots,\eta(\psi_{i-1}))\wedge\langle H\rangle(\psi_i) \quad \text{for } i=1,\ldots,n \tag{3.3}$$

So far we have that (3.3) holds for $i=n$. Assume that $i>1$ and (3.3) is valid for i. Then $\overline{T}\langle v_1,\ldots,v_i\rangle\langle v_1,\ldots,v_{i-1}\rangle$ and $\overline{\mathfrak{M}},\langle v_1,\ldots,v_{i-1}\rangle \Vdash D^{i-1}(\eta(\psi_1),\ldots,\eta(\psi_{i-1}))$. Expanding the definition of D we obtain $\overline{\mathfrak{M}},\langle v_1,\ldots,v_{i-1}\rangle \Vdash \langle T\rangle D^{i-2}(\eta(\psi_1),\ldots,\eta(\psi_{i-2}))\wedge\langle H\rangle(\psi_{i-1})$. (3.3) holds true and $\overline{\mathfrak{M}},\langle v_i\rangle \Vdash \psi_i$ which by the induction hypothesis implies $\mathfrak{M},v_i \Vdash \psi_i$. Finally, $\mathfrak{M},w \Vdash \langle R\rangle(\psi_1,\ldots,\psi_n)$. □

DEFINITION 3.2. *Let $ML(\Phi,\tau)$, $ML(\Phi,\tau')$ be modal languages. By interpretation of $ML(\Phi,\tau)$ in $ML(\Phi,\tau')$ we understand a mapping $(\cdot)^F : ML(\Phi,\tau)\longrightarrow ML(\Phi,\tau')$ for which the following holds ([2]):*

$q^F = p^F[q/p]$
$(\psi_1\wedge\psi_2)^F = (p_1\wedge p_2)^F[\psi_1^F/p_1,\ \psi_2^F/p_2]$
$(\neg\phi)^F = (\neg p)^F[\phi^F/p\ ]$;
$\langle R\rangle(\psi_1,\ldots,\psi_n)^F = \langle R\rangle(p_1,\ldots,p_n)^F[\psi_1^F/p_1,\ldots,\psi_n^F/p_n]$.

It is easy to show that η is an interpretation of $ML(\Phi,\tau)$ into $ML(\Phi,\bar{\tau})$.

Let Kr_n be the set of all formulas of the form:

A1). $\langle T\rangle\top \leftrightarrow \langle H\rangle\top,\ \langle H^{\sim}\rangle\top \to \langle H\rangle\bot$
A2). $\langle T\rangle\phi \to [T]\phi\ ,\quad \langle H\rangle\phi \to [H]\phi$
A3). $[H]^2\bot$

A4). $\phi \to [H]\langle H^{\sim}\rangle\phi$, $\phi \to [T]\langle T^{\sim}\rangle\phi$
$\phi \to [H^{\sim}]\langle H\rangle\phi$, $\phi \to [T^{\sim}]\langle T\rangle\phi$

A5). $\langle T\rangle\top \vee \langle T^{\sim}\rangle\top \to \bigvee_{\substack{k\geq 0\\ j\geq 0\\ k+j=n-1}} (\langle T\rangle^k\top \wedge [T]^{k+1}\bot \wedge \langle T^{\sim}\rangle^j\top \wedge [T^{\sim}]^{j+1}\bot)$,

A6). $\langle\overline{R}\rangle\phi \to \langle\overline{R}\rangle(\phi \wedge \langle T\rangle^{n-1}\top)$

A7). $\langle T\rangle^{n-1}\top \to \langle\overline{R}^{\sim}\rangle\top$

A8). $\phi \to [\overline{R}]\langle\overline{R}^{\sim}\rangle\phi$, $\phi \to [\overline{R}^{\sim}]\langle\overline{R}\rangle\phi$

A9). $\langle\overline{R}\rangle\phi \to [T]^k\langle\overline{R}\rangle\phi$, $[\overline{R}]\phi \to [T^{\sim}]^k[\overline{R}]\phi$ for $k < n$,

A10). $\phi \wedge [T]\psi \wedge [H]\chi \to [T][T^{\sim}]([H]\chi \to \phi)$
$\phi \wedge [T]\psi \wedge [H]\chi \to [H][H^{\sim}]([T]\psi \to \phi)$
(uniqueness of the pair).

Put $\overline{\mathsf{K}}_n$ for the class of frames of the form $\overline{\mathfrak{F}}$, where $\mathfrak{F} = (W, R)$ and $R \subseteq W^{n+1}$.

THEOREM 3.3. $\mathfrak{F} \in \mathbb{I}(\overline{\mathsf{K}}_n)$ *iff* $\mathfrak{F} \models Kr_n$.

PROOF: ($\Rightarrow$) it is an easy direction;
($\Leftarrow$) Assume that $\mathfrak{F} = (W, R, R^{\sim}, H^{\sim}, T, T^{\sim}) \models Kr_n$.
An abbreviation $\mathsf{el} := \langle T\rangle\top \vee \langle T^{\sim}\rangle\top \vee \langle H\rangle\top \vee \langle H^{\sim}\rangle\top$ will be used in the sequel. Let
$EL = \{w \in W : \mathfrak{F}, w \Vdash \mathsf{el}\} \cup \{w \in W : \mathfrak{F} \Vdash \langle\overline{R}\rangle\top\}$,
$EL^- = W/EL,\ AT^- = \{w \in W : \mathfrak{F}, w \Vdash \langle H\rangle\top\},\ AT = EL/AT^-$.
Obviously $W = AT \uplus AT^- \uplus EL^-$, where $\uplus$ stands for disjoint sum.
Let $W' = EL^- \cup AT$.
For all of the binary relations G we use the convention that G^0 is the identity on the domain of G.

Define $R'wv_1v_2\dots v_n$ iff
$w, v_1, v_2, \dots, v_n \in AT\ \&\ \exists_{w',v'\in W}[Rw'v'\ \&\ T^{n-1}v'v_1\ \&\ \bigwedge_{i=2}^{n} T^{n-i} \circ Hv'v_i$
$\&\ \bigvee_{i=0}^{n-1} T^i w'w]$.

Let us put $\mathfrak{G} = (W', R')$. We are going to show that
$\overline{\mathfrak{G}} = (\overline{W'}, \overline{R'}, \overline{R'}^{\sim}, H', H'^{\sim}, T', T'^{\sim}) \cong \mathfrak{F}$.

We put for convenience:
$\langle W'_{R'}\rangle := \{\langle w_1, w_2, \dots, w_k\rangle \in (W')^* : k > 1 \;\&\; \exists_{w_0, w_{k+1}, \dots, w_n \in W} R' w_0 w_1 \dots w_n\}$,
$\langle W'\rangle := \{\langle w\rangle : w \in W'\}$. Naturally $\overline{W'} = \langle W'_{R'}\rangle \uplus \langle W'\rangle$.

First note that A4). implies that Hwv iff $H^{\sim}vw$ and Twv iff $T^{\sim}vw$. These facts will be used in the sequel. We will show that

$$\text{for every } w_1, w_2 \in W\text{: if there exists } w \in W \text{ that fulfills } Tww_1 \;\&\; Hww_2\text{, then it is unique} \tag{3.4}$$

Let Tww_1 & Hww_2. Assume that $Tw'w_1$ & $Hw'w_2$ and define a valuation: $V(p_w) = \{w\},\ V(p_1) = \{w_1\},\ V(p_2) = \{w_2\}$.

Then by A10).: $\mathfrak{F}, V, w \Vdash p_w \wedge [T]p_1 \wedge [H]p_2 \to [T][T^{\sim}]([H]p_2 \to p_w)$. Therefore:
$\mathfrak{F}, V, w \Vdash [T][T^{\sim}]([H]p_2 \to p_w)$ and $\mathfrak{F}, V, w_1 \Vdash [T^{\sim}]([H]p_2 \to p_w)$. We obtain $\mathfrak{F}, V, w' \Vdash [H]p_2 \to p_w$ and finally $\mathfrak{F}, V, w' \Vdash p_w$, thus $w' = w$.

The required isomorphism $f : \overline{W'} \longrightarrow W$ is defined as follows:

- if $\langle w\rangle \in \langle W'\rangle$, then $f(\langle w\rangle) = w$
- Assume that $\langle w_1, w_2, \dots, w_k\rangle \in \langle \overline{W'_{R'}}\rangle$. By definition there can be found $w_0, \dots, w_{k+1}, \dots, w_n$ such that $R' w_0 w_1 w_2 \dots w_n$. Thus:
  $w_0, w_1, w_2, \dots, w_n \in AT$ and
  $\exists_{w', v' \in W}[Rw'v' \;\&\; T^{n-1}v'w_1 \;\&\; \bigwedge_{i=2}^{n} T^{n-i} \circ Hv'w_i \;\&\; \bigvee_{i=0}^{n-1} T^i w' w_0]$.
  So, let $Rw'_b v'_b \;\&\; T^{n-1}v'_b w_1 \;\&\; \bigwedge_{i=2}^{n} T^{n-i} \circ Hv'_b w_i \;\&\; \bigvee_{i=0}^{n-1} (T^i w'_b w_0)$
  According to A2). one can find v_2 for which $T^{n-2}v'_b v_2$, $Tv_2 w_1$ and $Hv_2 w_2$ hold. Since (3.4) v_2 is unique. Put $f(\langle w_1, w_2\rangle) = v_2$. Assume that $v_i = f(\langle w_1, \dots, w_i\rangle)$ is defined and $T^{n-i}v'_b v_i$. If $i < n$ then there exists v_{i+1} such that: $T^{n-(i+1)}v'_b v_{i+1}$. Due to the functionality of T (see A2).) we have $Tv_{i+1}v_i$ & $Hv_{i+1}w_{i+1}$. Obviously, v_{i+1} is unique and we can put $f(\langle w_1, \dots, w_{i+1}\rangle) = v_{i+1}$.
  Note that $f(\langle w_1, \dots, w_k\rangle)$ does not depend on the choice of $w_{k+1}, \dots, w_n$ such that $\langle w_1, \dots, w_n\rangle \in \overline{W'_{R'}}$, since we have built subsequently the values $f(\langle w_1, w_2\rangle), f(\langle w_1, w_2, w_3\rangle)$ etc.

It is clear that $f|\langle W'\rangle : \langle W'\rangle \longrightarrow AT \uplus EL^-$ in an injection.

Assume that $f(\langle w_1,\ldots,w_k\rangle) = f(\langle v_1,\ldots,v_j\rangle) = w$. It is not the case that $k \neq j$. For example, if $k < j$ then according to the definition of f: $T^{n-1}v'_a w_1$, $T^{n-k}v'_a w$ and $T^{n-1}v'_b v_1$, $T^{n-j}v'_b w$ for some v'_a, $v'_b \in W$. Then $T^{k-1}ww_1$ and $T^{j-1}wv_1$, but the last statement implies $T^{j-k}w_1v_1$ which contradicts $A1$)., $A2$). and $w_1 \in AT$.

We are going to show that $f(\langle w_1,\ldots,w_k\rangle) = f(\langle u_1,\ldots,u_k\rangle)$ implies $f(\langle w_1,\ldots,w_{k-1}\rangle) = f(\langle u_1,\ldots,u_{k-1}\rangle)$ and $w_k = u_k$.
Let $f(\langle w_1,\ldots,w_k\rangle) = f(\langle u_1,\ldots,u_k\rangle) = w$. Hww_k and Hwu_k implies that $w_k = u_k$. Assume that our hypothesis holds true and $f(\langle w_1,\ldots,w_i\rangle) = f(\langle u_1,\ldots,u_i\rangle)$ hold for $j \leq i \leq n$.
Then $Tf(\langle w_1,\ldots,w_j\rangle)f(\langle w_1,\ldots,w_{j-1}\rangle)$ and $Tf(\langle u_1,\ldots,u_j\rangle)f(\langle u_1,\ldots,u_{j-1}\rangle)$, so $f(\langle w_1,\ldots,w_{j-1}\rangle) = f(\langle u_1,\ldots,u_{j-1}\rangle)$. Moreover, by the definition of f: $Hf(\langle w_1,\ldots,w_{j-1}\rangle)w_{j-1}$ and $Hf(\langle u_1,\ldots,u_{j-1}\rangle)u_{j-1}$. Thus $w_{j-1} = u_{j-1}$.

Finally, f is 1-1 on the whole domain.

We have to show that f is a surjection. Let $w \in W$. If $w \in EL^- \cup AT$, then $f(\langle w\rangle) = w$. Otherwise $w \in AT^-$. According to A1). and A5). there are k_0, j_0 such that $k_0 + j_0 = n - 1$ and:

- there exists $w_1 \in AT$ such that $T^{k_0}ww_1$
- there exists $v \in W$ such that $(T^{\smile})^{j_0}wv$ and there is no $u \in W$ fulfilling Tuv

Moreover, let w_i be such that $T^{k_0-i+1} \circ Hww_i$ for $i = 2, \ldots k_0 + 1$ and $T^{k_0}ww_1$. We are going to prove that $\langle w_1,\ldots,w_{k_0+1}\rangle \in \langle\overline{W'_{R'}}\rangle$ and $f(\langle w_1,\ldots,$
$w_{k_0+1}\rangle) = w$.

Let $w_{k_0+2},\ldots,w_n$ fulfill $T^{n-i} \circ Hvw_i$ for $k_0 + 2 \leq i \leq n$. We can summarize:

$$\begin{gathered} T^{n-1}vw_1 \\ T^{n-i} \circ Hvw_i \text{ for } 2 \leq i \leq n \end{gathered} \tag{3.5}$$

Moreover, from A1). follows that $w_1, \ldots, w_n \in AT$. Since A7), there is $u \in W$ for which $R^{\sim}vu$ and by A8). Ruv. According to A9). there exists $w_0 \in AT$ such that $T^k u w_0$ and $Rw_0 v$ (where $k \in \{0, \ldots, n-1\}$). This fact together with (3.5) implies $R' w_0 w_1 w_2 \ldots w_n$, so $\langle w_1, \ldots, w_{k_0+1}\rangle \in \langle \overline{W'_{R'}}\rangle$ Since the definition of f and A2). $T^{n-k_0-1} f(\langle w_1 \ldots, w_n\rangle) w$ and due to the functionality of T we have $f(\langle w_1, \ldots, w_{k_0+1}\rangle) = w$.

We show that f preserves all the relations.

Assume that $\overline{T}\langle w_1, \ldots, w_n\rangle\langle w_1, \ldots, w_{n-1}\rangle$. Then according to the definition of f: $Tf(\langle w_1, \ldots, w_n\rangle) f(\langle w_1, \ldots, w_{n-1}\rangle)$.
If $Tf(x)f(y)$, then $x = \langle w_1, \ldots, w_k\rangle$ and $\overline{T'} x \langle w_1, \ldots, w_{k-1}\rangle$. According to the previous fact $Tf(x) f(\langle w_1, \ldots, w_{k-1}\rangle)$. But $f(\langle w_1, \ldots, w_{k-1}\rangle) = f(y)$ since T is a partial function. However f is also $1-1$, thus $y = \langle w_1, \ldots, w_{k-1}\rangle$. Finally – $\overline{T'}xy$.

$\overline{T'}^{\sim} xy$ implies $\overline{T'}yx$, $Tf(y)f(x)$ and $T^{\sim} f(x)f(y)$. Similarly like in the first case – $T^{\sim} f(x) f(y)$ implies $\overline{T'}^{\sim} xy$.

Also the definition of f allows us to note that $Hf(\langle w_1, \ldots, w_k\rangle) f(\langle w_k\rangle)$. Moreover, along the same lines it can be shown that:

- $Hf(x)f(y)$ implies $\overline{H'}xy$
- $\overline{H'}^{\sim} xy$ iff $H^{\sim} f(x) f(y)$

Assume that $\overline{R'}\langle u_1, \ldots, u_k\rangle\langle w_1, \ldots, w_n\rangle$. Then also $\overline{R'}\langle u_1\rangle\langle w_1, \ldots, w_n\rangle$ and:

$u_1, w_1, w_2, \ldots, w_n \in AT$ &
$\exists_{w', v' \in W} [Rw'v' \ \& \ T^{n-1} v' w_1 \ \& \ \bigwedge_{i=2}^{n} T^{n-i} \circ H v' w_i \ \& \ \bigvee_{i=0}^{n-1} T^i w' u_1]$.

Let

$$Rw'_a v'_a \ \& \ T^{n-1} v'_a w_1 \ \& \ \bigwedge_{i=2}^{n} T^{n-i} \circ H v'_a w_i \ \& \ \bigvee_{i=0}^{n-1} T^i w'_a u_1. \tag{3.6}$$

In virtue of A9). and $\bigvee_{i=0}^{n-1} T^i w'_a u_1$ we have

$$Ru_1 v_a'. \tag{3.7}$$

We are going to show by induction on n, that

$$\begin{array}{c}\text{from}\\ T^{n-1}v_a w_1 \ \& \ \bigwedge_{i=2}^{n} T^{n-i} \circ H v_a w_i \\ \text{follows} \\ v_a = f(\langle w_1, \dots, w_n \rangle)\end{array} \tag{3.8}$$

For $n = 2$: if $Tv_a w_1$ and $Hv_a w_2$, then from the definition of f: $v_a = f(\langle w_1, w_2 \rangle)$.

Let (3.8) holds for n and

$$T^n v_b w_1 \ \& \ \bigwedge_{i=2}^{n+1} T^{n+1-i} \circ H v_b w_i \tag{3.9}$$

or, equivalently

$$T^n v_b w_1 \ \& \ \bigwedge_{i=2}^{n} T^{n+1-i} \circ H v_b w_i \ \& \ H v_b w_{n+1} \tag{3.10}$$

Then for some $v_a \in W$:

$$Tv_b v_a \ \& \ T^{n-1} v_a w_1 \ \& \ T v_b v_a \ \& \ \bigwedge_{i=2}^{n} (T^{n-i} \circ H v_a w_i) \ \& \ H v_b w_{n+1} \tag{3.11}$$

due to the functionality of T. From the inductive assumption follows that $f(\langle w_1, \dots, w_n \rangle) = v_a$. Moreover, from $Tv_b v_a = f(\langle w_1, \dots, w_n \rangle)$, $Hv_b w_{n+1}$ and the definition of f we obtain $v_b = f(\langle w_1, \dots, w_{n+1} \rangle)$. Threrefore, we have proved (3.8). From (3.6) using (3.8) we obtain $v_a' = f(\langle w_1, \dots, w_n \rangle)$.
From (3.7) we obtain $Ru_1 f(\langle w_1, \dots, w_n \rangle)$. Now, using A9). it can be shown that $Rf(\langle u_1, \dots, u_k \rangle) f(\langle w_1, \dots, w_n \rangle)$.

Let $Rf(\langle u_1, \dots, u_k \rangle) f(\langle w_1, \dots, w_n \rangle)$. Now we can use the fact that f preserves T, H and write:

$$Rf(\langle u_1, \ldots, u_k\rangle)f(\langle w_1, \ldots, w_n\rangle) \ \&\ T^{n-1}f(\langle w_1, \ldots, w_n\rangle)w_1$$
$$\&\ \bigwedge_{i=2}^{n} T^{n-i} \circ Hf(\langle w_1, \ldots, w_n\rangle)w_i \ \&\ \bigvee_{i=0}^{n-1} T^i f(\langle u_1, \ldots, u_k\rangle)u_1$$

but the above means that

$$\exists_{w',v' \in W}[Rw'v' \ \&\ T^{n-1}v'w_1 \ \&\ \bigwedge_{i=2}^{n} T^{n-i} \circ Hv'w_i \ \&\ \bigvee_{i=0}^{n-1} T^i w'u_1]$$

and

$R'u_1w_1 \ldots w_n$. But the last statement implies $\overline{R'}\langle u_1, \ldots, u_k\rangle\langle w_1, \ldots, w_n\rangle$.

Finally $\overline{R'}^{\smile}xy$ iff $\overline{R'}yx$ iff $Rf(x)f(y)$ iff $R^{\smile}f(y)f(x)$. The first and the last equivalence follow from A8).

□

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