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FUNCTIONS DEFINABLE IN SOME FRAGMENTS OF SUGIHARA ALGEBRAS

This abstract was read to the Conference Logical Calculi, Wrocław November 1974.

I. The notion of a Sugihara matrix was introduced by J. M. Dunn in [1] as a generalisation of a matrix constructed by Sugihara in [2]. In this paper we are dealing only with algebraic properties of Sugihara matrices, so we need not to consider any element as being designated or not. We define a *Sugihara algebra* (SA) to be aby algebra $\underline{A} = (A, \sim, \wedge, \vee, \rightarrow)$. Where A is a chain, $\sim$ is a $1-1$ order inverting "onto" mapping on A with $\sim\sim a = a$, all $a \in A$ (an involution), and $\wedge, \vee, \rightarrow$ being defined as follows, for all $a, b \in A : a \wedge b = min(a, b), a \vee b = max(a, b)$, and

$$a \rightarrow b = \begin{cases} \sim a \vee b & \text{if } a \leqslant b, \\ \sim a \wedge b & \text{otherwise.} \end{cases}$$

We consider here two fragments of SA's: Define an *implicational Sugihara algebra* (ISA) to be any algebra $(A, \rightarrow)$, where $(A, \sim, \wedge, \vee, \rightarrow)$ is an SA for some $\sim, \wedge, \vee$. Define an *implicational-negational Sugihara algebra* ($INSA$) to be any algebra $(A, \sim, \rightarrow)$, where $(A, \sim, \wedge, \vee, \rightarrow)$ is an SA for some $\wedge, \vee$.

In this paper criteria for definability of functions in ISA's an $INSA$'s will be given.

By the *language* of ISA's (of $INSA$'s) we shall understand the sentential language $\underline{L}^i = (L^i, \rightarrow)$ ($\underline{L}^{in} = (L^{in}, \sim, \rightarrow)$), where L^i (L^{in}) is the set of formulas built up by means of sentential variables $p_1, p_2, \ldots$ and connective $\rightarrow$ (connectives $\sim, \rightarrow$). Let $\alpha = \alpha(p_{i_1}, \ldots, p_{i_n})$ be a formula of $\underline{L}^i$ ($\underline{L}^{in}$) containing but variables $p_{i_1}, \ldots, p_{i_n}$, let $\underline{A} = (A, \rightarrow)$ be an ISA (let $\underline{A} = (A, \sim, \rightarrow)$ be an $INSA$). If $a_1, \ldots, a_n \in A$ then the symbol

$\alpha(a_1, \ldots, a_n)$ stands for $h\alpha$ where h is any homomorphism of $\underline{L}^i$ (of $\underline{L}^{in}$) into $\underline{A}$ such that $hp_{i_1} = a_1, \ldots, hp_{i_n} = a_n$. Let f be a k-ary function on A, $f : A^k \to A$. Then f is said to be *definable* in the algebra $\underline{A}$ if there is a formula $\alpha(p_{i_1}, \ldots, p_{i_k})$ of $\underline{L}^i$ (of $\underline{L}^{in}$) such that for every $a_1, \ldots, a_k \in A$, $f(a_1, \ldots, a_k) = \alpha(a_1, \ldots, a_k)$. We shall say in such a case that α *defines* f in $\underline{A}$.

Let $\underline{A}$ be as above. For $a \in A$ we define $|a|$ to be a when $\sim a \leqslant a$ and $\sim a$ otherwise. If $\underline{a} = (a_1, \ldots, a_n)$ is a sequence of elements of A, then we put: $|\underline{a}| = (|a_1, \ldots, |a_n|)$, $max|\underline{a}| = max\{|a_1|, \ldots, |a_n|\}$. We will symbolically write $a \geqslant 0$ instead of $\sim a \leqslant a$ and $a < 0$ instead of $a <\sim a$.

II. In the whole Section II, $\underline{A} = (A, \to)$ stands for an arbitrary, but fixed, implicational Sugihara algebra and the variable f runs over all the k-ary functions on A, $f : A^k \to A$.

Definition 1. Two sequences $\underline{a} = (a_1, \ldots, a_k)$, $\underline{b} = (b_1, \ldots, b_k)$ of elements of A are *similar*, in symbols $\underline{a} \approx \underline{b}$, provided that the following conditions are satisfied:

(i) $|a_i| = max|\underline{a}|$ iff $|b_i| = max|\underline{b}|$, $i = 1, \ldots, k$,

(ii) if $|a_i| = max|\underline{a}|$ then $a_i < 0$ iff $b_i < 0$, $i = 1, \ldots, k$.

Definition 2. The function f *depends mainly* on i-th component if for every sequence $\underline{a} = (a_1, \ldots, a_k) \in A^k$ such that $a_i = max|\underline{a}|$, $f(\underline{a}) = a_i$.

Clearly, there are functions with no main component as well as functions with several main components.

Theorem 1. *A necessary and sufficient condition for f to be definable in $\underline{A}$ is that the following conditions hold for f:*

(1) for every $\underline{a} \in A^k$, $|f(\underline{a})| = max|\underline{a}|$,

(2) there is some i, $1 \leqslant i \leqslant k$, such that f depends mainly on i-th component,

(3) if $\underline{a}, \underline{b} \in A^k$ and $\underline{a} \approx \underline{b}$ then $f(\underline{a}) = \underline{a}(j)$ iff $f(\underline{b}) = \underline{b}(j)$, all j, $1 \leqslant j \leqslant k$.

III. In the Section III, $\underline{A} = (A, \sim, \to)$ stands for an arbitrary, but fixed, implicational-negational Sugihara algebra and the variable f runs over all the k-ary functions on A, $f : A^k \to A$.

Theorem 2. *A necessary and sufficient condition for f to be definable in $\underline{A}$ is that the following conditions hold for f:*

(1) for every $\underline{a} \in A^k$, $|f(\underline{a})| = max|\underline{a}|$,

(2) if $\underline{a}, \underline{b} \in A^k$ and $\underline{a} \approx \underline{b}$ then $f(\underline{a}) = \underline{a}(i)$ iff $f(\underline{b}) = \underline{b}(i)$, all i, $1 \leqslant i \leqslant k$.

References

[1] J. M. Dunn, *Algebraic completeness results for R-mingle and its extensions*, **The Journal of Symbolic Logic** 35 (1970), pp. 1–13.

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