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MATRIX REPRESENTATION FOR THE DUAL COUNTERPARTS OF ŁUKASIEWICZ n-VALUED SENTENTIAL CALCULI AND THE PROBLEM OF THEIR DEGREES OF MAXIMALITY

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1. Dual counterparts of Łukasiewicz n-valued calculi

Let $Ł_n = (L, C_n)$ be the n-valued Łukasiewicz sentential calculus (C_n is the consequence operation determined in L by the matrix $M_n = (\underline{A}_n, \{1\})$, where $\underline{A}_n$ denote the algebra formed by the set $A_n = \{0, {}^1/_{n-1}, {}^2/_{n-1}, \ldots, 1\}$ and the known Łukasiewicz operations $\rightarrow, \vee, \wedge, \sim$). By the *dual counterpart of the calculus* $Ł_n$ ($n \geqslant 2$) we shall understand the calculus $dŁ_n = (L, dC_n)$, where dC_n is the consequence operation dual with respect to C_n (see [4]).

Now, let us consider the following matrix:

$$\overline{M}_n = (\underline{A}_n, A_n - \{1\}). \tag{1}$$

Let us denote by $\overline{C}_n$ the consequence operation determined (in L) by $\overline{M}_n$.

Lemma 1. (cf. [2]) $dC_n = \overline{C}_n$.

From the above lemma it follows that $dŁ_n = (L, \overline{C}_n)$. In the sequel if we deal with $dŁ_n$ we shall always use this matrix characterization of the calculus $dŁ_n$.

In our later considerations we shall often use the connectives $\urcorner_n$ (of L) defined in [2]. We note that

$$\urcorner_n x = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

for $x \in A_n$. As a rule we shall always write $\urcorner$ instead of $\urcorner_n$ while the given n-valued calculus is considered. The following correspondence between the sets of the theorems of $Ł_n$ and $dŁ_n$ will be used in the paper

$$\urcorner\alpha \in dC_n(\emptyset) \text{ if and only if } \alpha \in C_n(\emptyset). \tag{3}$$

2. $d\mathbf{Ł}_n$-algebras

Given a consequence C defined on L, denote by $Matr(C)$, the class of all matrices M such that $C \leqslant Cn_M$. Let now $M = (A_M, I) \in Matr(dC_n)$. Put

$$a \approx_M b \text{ if and only if } \urcorner(a \to b), \urcorner(b \to a) \in I. \tag{4}$$

Now, we shall define some special subset of elements of the matrix M.

$$\bigvee_M = \{a \in A_M : \text{ there exists a formula } \alpha \in L, \text{ such that } \urcorner\alpha \in dC_n(\emptyset), \text{ and such that there exists valuation } h \text{ in } M \text{ for which } h\alpha = a\}. \tag{5}$$

Lemma 2.

(i) The relation $\approx_M$ is a congruence of A
(ii) $M/\approx_M$ is a one element set
(iii) Put $1_M = \bigvee^M/\approx_M$ (see (ii)). Then, we have: if $a \to b = 1_M$ and $b \to a = 1_M$ then $a = b$.

In the sequel we shall consider the class

$$Matr^R(dC_n) = \{{}^M/\approx_M : M \in Matr(dC_n)\}. \tag{6}$$

This class has similar properties to those of the class $Alg^R(C_n)$ of all S-algebras for $Ł_n$ in the sense of Rasiowa (cf. [3]). In particular, as it will be shown further, so called Lindenbaum matrix for dC_n is free in this class.

3. The equational characterization of $Matr^R(dC_n)$

Let $\underline{K}(\underline{A}_n)$ (given n, $n \geqslant 2$) denote the smallest equational class containing the algebra $\underline{A}_n$. Using the Birkhoff's characterization of equational classes we have

$$\underline{K}(\underline{A}_n) = H(S(P(\underline{A}_n))), \tag{7}$$

where P denotes the operation of taking direct products, S- of subalgebras and H- of homomorphic images. Let us also denote by $\underline{K}(M_n)$ the same class of algebras but treated as matrices with one distinguished element $1 = a \to a$ (a is any element of the algebra under consideration). Note, that this element is a natural counterpart of $1 \in A_n$.

LEMMA 3. (cf. [6]) $\underline{K}(M_n) = Alg^R(C_n)$.

Now, let us denote by $\underline{K}(\overline{M}_n)$ a class of matrices corresponding to $\underline{K}(\underline{A}_n)$ defined in the following manner

- i_1 If $A_M = A_n$, then $I_M = A_n - \{1\}$
- i_2 If A_M is a product of the indexed set of algebras $\{A_{M_i}\}_{i \in I}$, then $I_M = \prod\{I_{M_i} : i \in I\}$
- i_3 If A_M is a subalgebra of some algebra A_{M_0}, then $I_M = A_M \cap I_{M_0}$
- i_4 If $A_M = h(A_{M_0})$, where h is a homomorphism, then $I_M = h(I_{M_0})$.

From the representation theorem quoted in [6] as Lemma 5 we obtain

LEMMA 5. *For every matrix $M = (A_M, I_M) \in \underline{K}(\overline{M}_n)$ there exists a matrix $M' = (A_{M'}, I_M)$ with $A_{M'}$, being a subalgebra of $\underline{A}_n^T$ (T is a set of indices), such that $A_M \simeq A_{M'}$ and $Cn_M = Cn_{M'}$.*

LEMMA 6. *$M = (A_M, I_M) \in \underline{K}(\overline{M}_n)$ if and only if 1^0. $A_M \in \underline{K}(\underline{A}_n)$ and 2^0. $I_M = \{a \in A_M :=\!\mid a = 1_M\}$.*

Using thee results of [5] and the Lemmas 5 and 6 above one can easy prove that there holds

LEMMA 7. *$K(\overline{M}_n) \subseteq Matr^R(dC_n)$.*

Now, we are going to construct Lindenbaum matrix for $d\text{Ł}_n$. In the language L we introduce the following relation:

$$\alpha \approx \beta \text{ if and only if } \neg(\alpha \to \beta), \neg(\beta \to \alpha) \in dC_n(\emptyset). \tag{8}$$

One can easily verify that $\approx$ is a congruence relation on L. The quotient matrix

$$\Lambda_n = (^L/ \approx, dC_n(\emptyset)/ \approx) \tag{9}$$

will be called the Lindenbaum matrix for $d\text{Ł}_n$.

LEMMA 8. $\Lambda_n \in \underline{K}(\overline{M}_n)$.

H. Rasiowa in [3] proved that the Lindenbaum matrix for any consistent implicative calculus S is free in the class $Alg^R(S)$. This result is also valid for Łukasiewicz calculi.

The following lemma holds:

LEMMA 9. *Λ_n is free in the class $Matr^R(dC_n)$. The free generators of Λ_n are the classes determined by the sentential variables.*

The algebra which belongs to the equational class X and is free (in X) generators the whole class X. Thus the Lemmas 8, 9 give us the following

LEMMA 10. $Matr^R(dC_n) \subseteq \underline{K}(\overline{M}_n)$.

Combining the results of Lemma 6 and Lemma 7 we obtain the theorem being the main issue of the present section.

THEOREM 1. $Matr^R(dC_n) = \underline{K}(\overline{M}_n)$.

4. Degrees of maximality of $d\text{Ł}_n$

Let $M \in Matr^R(dC_n)$ be an arbitrary matrix and let $\approx_M$ be the congruence relation defined by (4). From the fact $\approx_M$ is a congruence of M (see Lemma 2(i)) it follows that

$$Cn_M = Cn_{M/\approx_M}. \tag{10}$$

As it was defined in the Section 2, the whole class of such $M/\approx_M$ is $Matr^R(dC_n)$. Moreover, according to the equational characterization given in Section 3 we have $Matr^R(dC_n) = \underline{K}(\overline{M}_n)$. This implies that the consequence operation determined by an arbitrary $M \in Matr(dC_n)$ is equal to some consequence determined by the matrix belonging to the class $\underline{K}(\overline{M}_n)$.

LEMMA 11. *Let $M = (A_M, I_M) \in \underline{K}(\overline{M}_n)$ and let $M_+ = (A_M, 1_M)$ be the corresponding matrix from $\underline{K}(M_n)$ (i.e. $1_M = \{\neg a : a \in I_M\}$ – see Lemma 6). Then for every $\alpha \in L$*

$$\alpha \in Cn_M(X) \text{ if and only if } \neg\alpha \in Cn_{M_+}(\neg X), \tag{11}$$

$$\neg\alpha \in Cn_M(\neg X) \text{ if and only if } \alpha \in Cn_{M_+}(X), \tag{12}$$

where $X \subseteq L$ and $\neg X$ denotes the set resulting from X by preceding each of its formulas by $\neg$.

LEMMA 12. (cf. [6]) *For each matrix $A \in \underline{K}(M_n)$ there are pairwise different submatrices $M_{m_1}, \ldots, M_{m_k}$ such that $Cn_A = Cn_{M_{m_1}} \times \ldots \times M_{m_k}$.*

From Lemma 1 and Lemma 12 by the argument used in §4 of [6] we obtain

THEOREM 2. *The degree of maximality of any n-valued logic dŁ$_n$ is finite.*

REMARK. The degrees of maximality of calculi Ł$_n$ and dŁ$_n$ are equal. In particular for the calculi Ł$_n$, dŁ$_n$ for which $(n-1)$ is prime, the degree of maximality equals 4 (cf. [1]).

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