Bulletin of the **S**ection of **L**ogic
Volume 4/1 (1975), pp. 33–36
reedition 2012 [original edition, pp. 33–37]

Tomasz Furmanowski

REMARKS ON DISCUSSIVE PROPOSITIONAL CALCULUS

This is an abstract of the paper which will be published in Studia Logica.

Jaśkowski has given a definition for system D_2 by interpretation in homogeneous predicate calculus of one variable see ([1], [2]). It is well known that homogeneous predicate calculus of one variable is equivalent to the modal system $S5$ of Lewis. Kotas in paper [3] and da Costa in paper [4] have shown that D_2 is finitely axiomatizable. In paper 4 the problem of construction discussive system based on a different modal system, for example $S4$, is given.

We shall show that for every modal system M such that $S4 \subseteq M \subseteq S5$ the discussive system $D(M)$ based on M is equal to D_2. It does seem not trivial because $S4 \neq S5$.

We shall use signs $\rightarrow, \Rightarrow, \vee, \wedge, \sim, M, L$ for the material implication, the strict implication, for disjunction, conjunction, negation, possibility and necessity, respectively. Logical system we regard as the set of formulas. L and $s(L)$ are the set of thesis and the set of well formed formulas for a given system L, respectively. $P, Q, R, \ldots$ and so on are the signs for formulas and $p, q, r, \ldots$ and so are the signs for sentential variables.

Now we define system $M - S4$ (see [3]).

Definition 1.

a) $s(M - S4) = s(S4)$.
b) $P \in M - S4$ iff $MP \in S4$.

Lemma 1. *The following axiom schemes, rules of inferences and definitions constitute a complete axiomatization of $M - S4$:*

I. Axiom schemes:

1. $L(P \to (\sim P \to Q))$
2. $L((P \to Q) \to [(Q \to R) \to (P \to R)])$
3. $L[(\sim P \to P) \to P]$
4. $L(LP \to P)$
5. $L(L(P \to Q) \to (LP \to LQ))$
6. $L(LP \to LLP)$

II. Derivation rules:

$(r_1)\ \dfrac{LP,\ L(P \to Q)}{LQ}$ $(r_3)\ \dfrac{LP}{P}$

$(r_2)\ \dfrac{LP}{LLP}$ $(r_4)\ \dfrac{MP}{P}$

III. Definitions:

1. $P \wedge Q = \sim(\sim P \vee \sim Q)$
2. $P \leftrightarrow Q = (P \to Q) \wedge (Q \to P)$
3. $P \Rightarrow Q = L(P \to Q)$
4. $MP = \sim L \sim P.$
5. $P \Leftrightarrow Q = L(P \leftrightarrow Q).$

Lemma 2. *Formula $L(MP \to LMP)$ is a thesis of system $M - S4$.*

System $M - S5$ arises from $S5$ in the same way as $M - S4$ from $S4$. It is easy to see (cf. [3]) that the sets of axioms and rules for $M - S5$ are the same as for $M - S4$ except of axiom number 6. In the system $M - S5$ as the sixth axiom is the formula $L(MP \to LMP)$. Thus from Lemmas 1, 2 we have $M - S5 \subset M - S4$. The converse inclusion is trivial.

Theorem 1. *$P \in M - S4$ iff $P \in M - S5$.*

Corollary 1. *For every modal system M such that $S4 \subseteq M \subseteq S5$ we have: $MP \in M$ iff $MP \in S5$.*

Following Jaśkowski we shall give a definition for the discussive system $D(M)$ based on a modal system M.

Definition 2. Connectives determined as follows:

a. $P \rightarrow_d Q = MP \rightarrow Q$,
b. $P \wedge_d Q = P \wedge MQ$,

we shall name a discussive implication and discussive conjunction, respectively.

Definition 3. By discussive system $D(M)$ based on modal system M we mean:

a. 1^o If $p, q, r, \ldots$ are the sentential variables then $p, q, r, \ldots \in s(D(M))$.
2^o If $P, Q \in s(D(M))$ then $P \vee Q$, $P \wedge Q$, $P \rightarrow_d Q$, $P \wedge_d Q$, $\sim P \in s(D(M))$.
3^o $s(D(M))$ is the smallest set satisfying the conditions 1^o and 2^o.

b. $P \in D(M)$ iff $MP' \in M$ where P' is the formula obtained from P by elimination the symbols $\rightarrow_d$ and $\wedge_d$ according to their definitions.

In this notation $D(S5) = D_2$. The following theorem results from the Corollary 1 and Definition 3:

Theorem 2. *For every modal system M such that $S4 \subseteq M \subseteq S5$, $D(M) = D_2$.*

In this way there arises the problem: For what different modal system M, $D(M) = D_2$?

References

[1] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, **Studia Soc. Sci. Torunensis**, vol. I, nr 5 (1948), pp. 57–77.

[2] S. Jaśkowski, *O konjunkcji dyskusyjnej w rachunku zdań dla systemów dedukcyjnych sprzecznych*, **Studia Soc. Sci. Torunensis**, vol. I, nr 8 (1959), pp. 171–172.

[3] J. Kotas, *The axiomatization of S. Jaśkowski discussive logic*, forthcoming in **Studia Logica**.

[4] N. C. A. da Costa, *Remarks on Jaśkowski discussive logic*, **Forthcoming in Reports on Mathematical Logic** 4 (1975).

[5] Feys, **Modal Logic**, 1965.

[6] H. Rasiowa, R. Sikorski, **The Mathematics of Metamathematics**, Warszawa, 1968.

[7] J. Kotas, A. Pieczkowski, *Allgemeine Logische und Mathematische Theorien*, **Zeitschrift Mathematische Logik und Grundlagen der Mathematik**, Bd. 16 (1970), pp. 353–376.

Institute of Mathematics
Nicholas Copernicus University
Toruń