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ON HOMOGENOUS FRAGMENTS OF NORMAL MODAL PROPOSITIONAL LOGICS

This is an abstract of the lecture presented in November 1974 at the VI Conference on Logical Calculi, Wrocław. The full text with detailed proofs will appear in No. 5 of Reports on Mathematical Logic.

The purpose of the present paper is to discuss the L-fragments and M-fragments of normal modal propositional logic, where by L-fragment (resp. M-fragment) of the modal, logic $\underline{P}$ we mean the set of all theorems belonging to $\underline{P}$, whose first symbol is L (resp. M).

The paper is divided into three sections. In §1 we describe notations and terminology used in the sequel and we define some notations which are basic for our paper. In §2 for some classes of axiomatizable normal calculi we axiomatize their L- and M-counterparts. Finally, in §3 the fragments of some modal normal logics are investigated.

§1. We use the well-known logical and set-theoretical notation. Particularly, ω denotes the set of natural numbers; while k, l, m, n are used as symbols denoting natural numbers; $\sim, \rightarrow, L, M$ represents the logical connectives and denotes negation, material implication, necessity and possibility, respectively. The formulas will be represented by capitals $A, B, \ldots$, By FOR we denote the set of all formulas, while latin capitals $X, Y, \ldots$ are used as variables running over sets of formulas. Underlined capitals denote the propositional calculi and finally script capitals will represent sets of propositional logics. We put $L^0A = A$, $L^{n+1}A = LL^nA$. The abbreviation M^nA is defined in the same way. By $\bigvee\limits_{i \leqslant k} A_i$ we mean the generalized disjunction with $k + 1$-disjuncts.

As usual by detachment we understand $\{(A, A \to B, B) : A, B \in FOR\}$, whereas strict detachment is defined as $\{(A, L(A \to B), B) : A, B \in FOR\}$. For any n, m we put $RM^n_m = \{(M^n A, M^m A) : A \in FOR\}$ and $RL^n_m = \{(L^n A, L^m A) : A \in FOR\}$. For the sake of simplicity we shall write RM^+, RM^-, RL^+, RL^- instead of RM^0_1, RM^1_0, RL^0_1, RL^1_0 respectively. The RL^+ is the well-known Gödel's rule, while RM^- will be called the rule of Jaśkowski. Sb denotes the rule of substitution.

Let $\underline{CL}$ denote the set of all classical tautologies. C is the consequence operator defined by the $\underline{CL}$ and detachment, whereas Cn is defined by means of $\underline{CL}$, detachment and RL^+.

We list of some well-known normal calculi which will be investigated in the sequel

$$
\begin{array}{lcl}
\underline{K} & = & Cn(L(A \to B) \to (LA \to LB)) \\
\underline{TR} & = & Cn(\underline{K}, LA \equiv A) \\
\underline{VER} & = & Cn(\underline{K}, LA) \\
\underline{D} & = & Cn(\underline{K}, M(A \to A)) \\
\underline{D}^* & = & Cn(\underline{D}, LA \to MLA) \\
\underline{T} & = & Cn(\underline{K}, LA \to A) \\
\underline{BK} & = & Cn(\underline{T}, MLA \to A) \\
\underline{S4} & = & Cn(\underline{T}, LA \to LLA) \\
\underline{S5} & = & Cn(\underline{T}, MLA \to LA) \\
\underline{S4K.1} & = & Cn(\underline{S4}, LMA \to MLA)
\end{array}
$$

We put $\mathcal{L} = \{X : \underline{K} \subset SbC(X) = X\}$, and $\mathcal{N} = \{X : \underline{K} \subset SbCn(X)\}$. The elements of $\mathcal{L}$ will be called logics, while by normal logics we understand the elements of $\mathcal{N}$.

Let us put

$$
\begin{array}{lcl}
M^n(X) & = & \{A : M^n A \in X\} \\
L^n(X) & = & \{A : L^n A \in X\} \\
(X)M^n & = & \{M^n A \in X : A \in FOR\} \\
(X)L^n & = & \{L^n A \in X : A \in FOR\} \\
M^\omega(X) & = & \bigcup_{n\in\omega} M^n(X) \\
L^\omega(X) & = & \bigcup_{n\in\omega} L^n(X) \\
(X)M^\omega & = & \bigcup_{n\in\omega} (X)M^n \\
(X)L^\omega & = & \bigcup_{n\in\omega} (X)L^n
\end{array}
$$

For brevity's sake we shall write $M(X), L(X), (X)M, (X)L$ instead of $M^1(X), L^1(X), (X)M^1, (X)L^1$ respectively. The $M(X)$ is called the M-counterpart of X, while $(X)M$ is the M-fragment of X. Analogously for $L(X), (X)L$. L-fragments or M-fragments of will be called homogenous fragments of X. It will be seen that investigation of fragments is equivalent to study of counterparts (comp. Fact below).

We begin with the following simple but useful observations

(1.1)	$X = M^0(X) = L^0(X)$		
(1.2)	$M^n(X) \subset M^m(X)$	iff	X is closed on the RM^n_m
(1.3)	$L^n(X) \subset L^m(X)$	iff	X is closed on the RL^n_m
(1.4)	$L^n(X) = X$	iff	X is closed on the RL^0_n and RL^n_0
(1.5)	$M^n(X) = X$	iff	X is closed on the RM^0_n and RM^n_0
(1.6)	If $X \subset Y$	then	$M^n(X) \subset M^n(Y), L^n(X) \subset L^n(Y)$
			$(X)M^n \subset (Y)M^n, (X)L^n \subset (Y)L^n$
(1.7)	$M^n((X)M^n) = M^n(X)$	and	$L^n((X)L^n) = L^n(X)$

FACT 1.

(i) $L^n(X) = L^n(Y)$ iff $(X)L^n = (Y)L^n$

(ii) $M^n(X) = M^n(Y)$ iff $(X)M^n = (Y)M^n$.

§2. First, we axiomatize the $M^n(\underline{P})$ for every normal and axiomatizable $\underline{P}$, which is closed on the RM^{2n}_n, $R1$ and $R2$ defined below and contains $\underline{D}$. We adopt the method of [1] due to D. Makinson.

Let us put

$$
\begin{aligned}
R1 &= \{(M^nLA, M^nLLA) : A \in FOR\}\\
R2 &= \{(M^nLM^n, M^{2n}A) : A \in FOR\}\\
R3 &= \{(M^nA, M^nL(A \to B), M^nB) : A, B \in FOR\}\\
R4 &= \{(M^nL(A \to B), M^nL(LA \to LB)) : A, B \in FOR\}\\
R5 &= \{(LM^nA, M^nA) : A \in FOR\}\\
R6 &= \{(L(A \to B), L(LA \to LB)) : A, B \in FOR\}.
\end{aligned}
$$

LEMMA 1. *$\underline{P}$ is closed on the $R3$ and $R4$.*

Let Cm^n be a consequence operator determinate by strict detachment, $R5$, $R6$ and RM^n_0. $\underline{P}$ is axiomatizable, i.e. there is Ap such that $\underline{P} = Cn(Ap)$.

LEMMA 2. *$Cm^n(LAp) \subset M^n(\underline{P})$.*

LEMMA 3. $\underline{P} \subset L(Cm^n(LAp))$.

THEOREM 1. $M^n(\underline{P}) = Cm^n(LAp)$.

Secondly, we are axiomatizing $L^n(\underline{P})$ for every normal and axiomatizable $\underline{P}$ which is closed on the RL^{2n}_n.

LEMMA 4. *For every* n *if* $\underline{P}$ *is normal then* $L^n(\underline{P})$ *is normal.*

Let Ap be the axiom of $\underline{P}$. Cn^n is the consequence operator defined by detachment, RL^+ and RL^n_0.

THEOREM 2. $L^n(\underline{P}) = Cn^n(Ap)$.

§3. In this section we investigate the homogenous parts of particular normal calculi.

Unless stated explicitely otherwise we assume that $\underline{P} \in \mathcal{N}$.

Let $m \geqslant 1$, $n \geqslant 0$. First we note that

(3.1) $L^m(\underline{VER}) = FOR$ and $M^m(\underline{VER}) = \emptyset$
(3.2) $L^n(\underline{TR}) = \underline{TR} = M^n(\underline{TR})$
(3.3) If $\underline{P} \subset \underline{VER}$ then $M^m(\underline{P}) = \emptyset$
(3.4) $M^m(\underline{K}) = \emptyset, M^\omega(\underline{K}) = \underline{K}$.

for every $m \geq n \geq 0$ the following hold
(3.5) $L^n(\underline{P}) \subset L^m(\underline{P}) \subset L^\omega(\underline{P})$
(3.6) If $\underline{T} \subset \underline{P}$ then $\underline{P} = L^n(\underline{P}) = L^\omega(\underline{P})$
(3.7) If $\underline{D} \subset \underline{P}$ then $M^n(\underline{P}) \subset M^m(\underline{P})$
(3.8) $\underline{K} = L^n(\underline{K}) = L^\omega(\underline{K})$
(3.9) $\underline{D} = L^n(\underline{D}) = L^\omega(\underline{D})$
(3.10) $\underline{D} = M^n(\underline{D}) = M^\omega(\underline{D})$.

It is well-known that for every $n \geqslant 1$

(3.11) If $\underline{S4} \subset \underline{P}$ then $M(\underline{P}) = M^n(\underline{P}) = M^\omega(\underline{P})$
(3.12) $M(\underline{T}) \not\subseteq M^2(\underline{T})$ and $M(\underline{BK}) \not\subseteq M^2(BK)$.

For every $\underline{P}$, such that $\underline{T} \subset \underline{P}$ we have the following characterization of $M^\omega(\underline{P})$, which to the, author of the present paper seems to be the formal analog of the well-known Gödel's interpretation of possibility:

THEOREM 3. $A \in M^\omega(\underline{P})$ *iff* $Cn(\underline{P} \cup \{\sim A\}) = FOR$.

It should be observed that the assumption $\underline{T} \subset \underline{P}$ is essential. However, for every normal $\underline{P}$ we may establish the following weaker version of Theorem 3.

THEOREM 4. *$Cn(\underline{P} \cup \{\sim A\}) = FOR$ iff there exists K such that $\bigvee_{i \leqslant k} M^i A \in \underline{P}$.*

Now we are going to discuss in detail the relation between $\underline{P}$ and $M(\underline{P})$.

THEOREM 5. *$\underline{P} \subset M(\underline{P})$ iff $\underline{D} \subset \underline{P}$ iff $M(\underline{P}) \neq \emptyset$.*

The discussion when $M(\underline{P}) \subset \underline{P}$ we divide into two parts.
First, the case when $M(\underline{P}) = \emptyset$ will be considered

THEOREM 6. *$M(\underline{P}) = \emptyset$ iff $\underline{P} \subset \underline{VER}$.*

Secondly, we investigate the case $M(\underline{P}) \neq \emptyset$, i.e. by Theorem 5 when $\underline{D} \subset \underline{P}$.

(3.13) If $\underline{T} \subset \underline{P} \subsetneq \underline{TR}$ then $\underline{P} \subsetneq M(\underline{P})$.

Let $\underline{S}$ be the smallest logic in the family of all Scroggs logics which are different from $\underline{TR}$.

(3.14) If $\underline{D}^* \subset \underline{P} \subset \underline{S}$ then $M(\underline{P})$ is not closed on the RL^+ and the detachment.

But fortunately

(3.15) If $\underline{D} \subset \underline{P}$ and $\underline{P}$ is closed on the RM_1^2 then $M(\underline{P})$ is closed on the strict detachment

Let us recall the result of Furmanowski [2]:

(3.16) $M(\underline{S4}) = M(\underline{S5})$.

Obviously there are normal logics containing $\underline{S4}$, such that their M-counterparts are different from $M(\underline{S4})$. The Scroggs logics may serve as trivial examples, whereas the non-trivial example of such logics is McKinsey's calculus $\underline{S4K.1}$. Namely,

(3.17) $M(\underline{S4K.1}) = \underline{TR}$

Combining (3.17) with Theorem 3 we may obtain the following characterization of $\underline{S4K.1}$:

(3.18) $\underline{S4K.1}$ is smallest logic in the family of normal logics $\underline{P}$ such that $\underline{S4} \subset \underline{P}$ and $\underline{TR} = \{A : Cn(\underline{P} \cup \{\sim A\}) = FOR$

Let us point out that $\underline{BK}$ have the following property which contrast with discussed above (3.16) and (3.17).

(3.19) For an $\underline{P}, \underline{P}'$ such that $\underline{BK} \subset \underline{P} \cap \underline{P}'$
$\underline{P} = \underline{P}'$ iff $M(\underline{P}) = M(\underline{P}')$.

We end our consideration by claim that no L- and M- counterparts of normal logic contained in $\underline{S5}$ have the finite characteristic matrix.

References

[1] N. C. A. da Costa, *Remarks on Jaśkowski's discussive logic*, **Reports on Mathematical Logic** 4 (1975).

[2] T. Furmanowski, *Remarks on discussive propositional calculus*, forthcoming in **Studia Logica**.

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