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A REPRESENTATION THEOREM FOR THE LATTICE OF STANDARD CONSEQUENCE OPERATIONS

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§1. Introduction

A *sentential language* S is an algebra of finite type (i.e. a set equipped with a finite number of finitary functions) which is absolutely free in the class of all similar algebras, freely generated by a set of "(sentential) variables". A consequence operation on S is a function C from the power set of S, PS, into PS satisfying the following three conditions:

i) $X \subseteq C(X)$, all $X \subseteq S$;
ii) $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$, all $X, Y \subseteq S$;
iii) $C(C(X)) \subseteq C(X)$, all $X \subseteq S$.

A consequence operation C is algebraic (or finite) if

iv) $C(X) = U\{C(Y)|Y\ X, Y \text{ finite}\}$

and is *structural* if for all endomorphisms h of S,

$$hC(X) \subseteq C(h(X)), \text{ all } X \subseteq S.$$

If C is both algebraic and structural, C is called *standard* (see [W] and [B]).

It is known that the collection ST (for "standard") of all standard consequence operations on S forms a complete lattice when the ordering is defined by:

$$C \leqslant C' \text{ if } C(X) \subseteq C'(X), \text{ all } X \subseteq S.$$

Indeed, if C_i, $i \in I$ is a subset of ST, then $\bigvee(C_i, i \in I)$ is the consequence operation C satisfying:

$$\alpha \in C(X) \text{ iff, for some } i_1, \ldots, i_n \in I,$$

$$\alpha \in C_{i_1}(C_{i_2}(\ldots(C_{i_n}(X)))\ldots)$$

In this note, a representation of the lattice ST will be given.

For any sentential language S, let $L(S)$ be the first-order language (without equality) having one unary predicate symbol T, whose *terms* form an algebra isomorphic to S. (Thus we identify the formulas of S with the terms of $L(S)$). Let H denote the collection of all sentences of $L(S)$ which are the *universal closures* of formulas of the form

$$(*) \qquad T(\tau_1) \wedge T(\tau_2) \wedge \ldots \wedge T(\tau_k). \to T(\tau)$$

where $k \geqslant 0$, τ_i, τ are terms. When $k = 0$ $(*)$ becomes just $T(\tau)$. H is thus the collection of (strict) basic Horn sentences (see [CK]).

For a subset Γ of H, let $\overline{\Gamma} = \{\sigma \in H | \Gamma \models \sigma\}$, where is the classical logical consequence operation. It is easily seen that the collection HN (for "Horn") of subsets of H of the form $\overline{\Gamma}$, $\Gamma \subseteq H$, is a complete lattice when ordered by set inclusion. Indeed, if $\overline{\Gamma}_i$, $i = I$ are in HN, then

$$\Lambda(\Gamma_i, i \in I) = \bigcap(\overline{\Gamma}_i : i \in I),$$

i.e. the meet operation in HN is just intersection.

In the next section, we will prove the following

Theorem. *The lattices ST and HN are isomorphic.*

Corollary. *ST is a complete, compactly generated ("algebraic") lattice (i.e. every element is a join of compact elements).*

Indeed, the lattice HN is algebraic, since $\overline{\Gamma} = \bigvee(\overline{\Gamma}_f : \Gamma_f \subseteq \Gamma$, Γ_f finite), and the sets $\overline{\Gamma}_f$, with Γ_f finite are compact in HN.

§2. Proof of the theorem

A matrix M is a pair (A, T) consisting of an algebra A similar to S and a subset T of A (we use the same letter for an algebra and its underlying set.)

Equivalently, a matrix is just a $L(S)$-structure. Any matrix $M = (A, T)$ determines a structural consequence operation C_M on S by: for $X \subseteq S$, $\tau \in S$,

$\tau \in C_M(X)$ if, for any homomorphism $h : S \to A$, $h(\tau) \in T$ whenever $h(X) \subseteq T$.

For any consequence operation C on S, let $K(C)$ be the class of all matrices M such that $C \leqslant C_M$.

Lemma 1. *A class K of matrices is $K(C)$ for some standard C iff $K = Mod\overline{\Gamma}$, for some $\Gamma \subseteq H$.*

Proof. If $K = K(C)$, C standard, then by ([B], Theorem 2.6) it follows that $K = Mod\overline{\Gamma}$, where Γ consists of all sentences

$$(**) \qquad \forall \overrightarrow{x}[T(\tau_1)\ldots T(\tau_k). \to T(\tau)]$$

such that $\tau \in C(\tau_1, \ldots, \tau_k)$.

Conversely, if $K = Mod\overline{\Gamma}$, define C by:

$\tau \in C(X)$ iff $\tau \in C_M(X)$, all $M \in K$. By ([B], Theorem 2.9) C is standard, and it is easily seen that $K = K(C)$.

Remark. It is well-known [G] that an axiomatizable class K of matrices is $Mod\overline{\Gamma}$, some $\Gamma \subseteq H$ iff K is closed under arbitrary products and substructures.

Lemma 2. *Suppose C_i are standard consequence operations on S, Γ_i are subsets of H such that $K(C_i) = Mod\overline{\Gamma}_i$, $i = 1, 2$. Then $C_1 \leqslant C_2$ iff $\overline{\Gamma_1} \subseteq \overline{\Gamma}_2$.*

Proof. By (W], Theorem 3.1) it follows that $C_1 \leqslant C_2$ iff $K(C_2) \subseteq K(C_1)$. The lemma thus follows easily.

From Lemmas 1, 2 it follows that the function $C \mapsto \Gamma_C$ taking the standard consequence C to the subset $\Gamma_C = \overline{\Gamma}_C$ of H with $K(C) = Mod\Gamma_C$ is a lattice isomorphism $ST \to HN$. Notice that the sets $\overline{\Gamma}$, with Γ finite, correspond to the "finitely based" [B] consequence operations on S, i.e. those definable from a finite number of structural rules [W]. These consequence operations are precisely the compact elements of the lattice ST.

We close with several problems. It is known that if M is a finite matrix, C_M is standard, but we do not know whether C_M is compact on ST. Further, we don't know if the meet of two compact elements is compact.

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