Bulletin of the **S**ection of **L**ogic
Volume 4/2 (1975), pp. 57–62

reedition 2012 [original edition, pp. 57–64]

Jerzy J. Błaszczuk, Wiesław Dziobiak

REMARKS ON PERZANOWSKI'S MODAL SYSTEM

This paper was presented at the Seminar of the Section of Logic, Institute of Mathematics Nicholas Copernicus University, held by Professor Jerzy Kotas, Toruń, March 1975. An altered version of the paper will be published in Studia Logica.

S. Jaśkowski in [3] and [4] defined by interpretation a new logic system D_2 called "discussive system". S. Jaśkowski based his construction on the Lewis system $S5$. Applying Jaśkowski's method it is possible to construct discussive systems taking other modal systems as the base.

T. Furmanowski in [2] proved that the discussive system based on the Lewis system $S4$, as well as on any other intermediate system between $S4$ and $S5$, is equal to the discussive system D_2.

So, there arises the problem of finding a minimal modal system, over which the discussive system is equal to D_2. This problem and its solution has been suggested by J. Perzanowski (note in [2]). Defining the minimal system in the class of normal systems (i.e. closed on the substitution, detachment and Godel's rules), contained between T (Feys-von Wright modal system) and $S4$, he adds to the set of axioms of the system T the two following axioms: $MLCLpLLp$, $MLCMLpLp$ and the rule: if $MM\alpha$, then $M\alpha$.

The aim of this paper is to prove that axioms given by J. Perzanowski (i.e. $MLCLpLLp$, $MLCMLpLp$) may be omitted and to localize the so received system in relation to other systems; in particular we show that this system is essentially different from the system $S4$.

Throughout this paper $\underline{R}$ will denote the real straight line with natural topology given by the closure operation C and $\setminus$, $\cup$ are the signs of set-theoretical operations of complementation and sum. In this paper we use the notation of Łukasiewicz. The symbols C, N, L, M, denote material implication, negation, necessity and possibility, respectively. Logic systems will be treated as sets of formulas.

The symbol $\underline{K}$ will denote the class of all modal normal systems which are intermediate between T and $S4$. If $S \in \underline{K}$ then the symbol $M - S$ denotes the set of all formulas of the systems S, which become theses of S when preceded by the sign M. Since logic systems are treated as sets of formulas, $M - S$ is a logic system.

Let $\underline{A}$ be the set consisting of the following formulas:

(A_1) $CCpqCCqrCpr$,
(A_2) $CpCNpq$,
(A_3) $CCNppp$,
(A_4) $CLpp$,
(A_5) $CLCpqCLpLq$.

In further considerations we shall employ the following rules of deduction:

(R_1) substitution rule,
(R_2) if α and $C\alpha\beta$, then β,
(R_3) if α, then $L\alpha$,
(R_4) if $MM\alpha$, then $M\alpha$,
(R_5) if $L\alpha$ and $LC\alpha\beta$, then $L\alpha\beta$,
(R_6) if $L\alpha$, then $LL\alpha$,
(R_7) if $LMM\alpha$, then $LM\alpha$,
(R_8) if $L\alpha$, then α,
(R_9) if $M\alpha$, then α.

The symbol $Cn(\underline{A}; R_1, \dots, R_n)$ denotes the set of all formulas which are consequences of the set $\underline{A}$ with respect to the rules of deduction $(R_1), \dots, (R_n)$.

Definition 1. $T^* = Cn(\underline{A}, R_1, R_2, R_3, R_4)$.

Remark. The system resulting from T^* by canceling the rule (R_4) and adding to the set $\underline{A}$ one of the formulas: $CLpMLLp$ or $CLMMpMp$ is deductively equivalent to T^*.

Directly from the definition of the system T^* it follows that $T^* \in K$ and $T \neq T^*$.

THEOREM 1. $T^* \neq S4$.

PROOF. To show that $T^* \neq S4$ it is enough to construct T-matrix $\underline{U}$ and show that $CLpLLp \notin E(\underline{U})$. Applying the operation C of the topological space $\underline{R}$ we define an operation $C_1 : 2^{\underline{R}} \to 2^{\underline{R}}$ in the following way:

$$C_1 A = \begin{cases} \emptyset, & \text{when} \quad A = \emptyset \\ CA \cup (0,1), & \text{when} \quad A \neq \emptyset \end{cases}$$

where $\emptyset$ is an empty set.

Let $\underline{U} = \langle 2^{\underline{R}}; \{\underline{R}\}, \to, \sim, I \rangle$, where $\sim A = \underline{R} \backslash A$; $A \to B = \sim A \cup B$, $IA = \sim C_1 \sim A$. It is easy to verify that the axioms $(A_1), (A_2), (A_3), (A_4), (A_5)$ belong to $E(\underline{U})$ and the rules $(R_1), (R_2), (R_3)$ are permissible in the matrix $\underline{U}$. Let $MM\alpha \in E(\underline{U})$ i.e. for each valuation e in the matrix $\underline{U}$ we get $h^e(MM\alpha) = \underline{R}$. Then $h^e(MM\alpha) = C_1(C_1 h^e(\alpha)) = C_1(Ch^e(\alpha) \cup (0,1)) = Ch^e(\alpha) \cup (0,1) = \underline{R}$. From connectedness of the set $(0,1)$ we get $Ch^e(\alpha) \cup (0,1) = \underline{R}$.
Since $Ch^e(\alpha) \cup (0,1) = h^e(M\alpha)$, then $M\alpha \in E(\underline{U})$.

So, we have verified that $\underline{U}$ is a T^*-matrix. To prove that $CLpLLp \notin E(\underline{U})$ it is sufficient to substitute the value $(0, +\infty)$ of the matrix $\underline{U}$ for the variable p.

Let $\underline{B}$ be the set consisting of the following formulas:

(LA_1) $LCCpqCCqrCpr$,
(LA_2) $LCpCNpq$,
(LA_3) $LCCNppp$,
(LA_4) $LCLpp$,
(LA_5) $LCLCpqCLpLq$.

LEMMA 1.

i) $Cn(\underline{B}; R_1, R_5, R_6, R_7, R_8) = T^*$
ii) $Cn(\underline{B}; R_1, R_5, R_6, R_7, R_8, R_9) = M - T^*$.

PROOF. We shall prove only that the rules $(R_5), (R_7)$ are permissible in the system $M - T^*$. The further part of this proof is similar to the one of the Lemmas 2, 3 given in [5] and relating to the system $M - S5$. For an arbitrary logic system S by $\alpha \in S$ we mean that formula α is a thesis of the system S.

(R_5): Let L, $LC\alpha\beta \in M - T^*$. It follows that $ML\alpha$, $MLC\alpha\beta \in T^*$. It is easy to see that $CLpMLLp \in T^*$. Since $MMCpLLp \in T$ and applying

(R_4) we get $MCpLLp \in T^*$. Then, $CLpMLLp \in T^*$. Now we shall give a sequence of formulas which is an outline of a proof of the formula $ML\beta$ in the system T^*.

$CLpMLLp \in T^*$
$CLC\alpha\beta MLLC\alpha\beta \in T^*$
$CMLC\alpha\beta MMLLC\alpha\beta \in T^*$
$MMLLC\alpha\beta \in T^*$
$CLAN\alpha\beta AL\beta MN\alpha \in T^*$
$CLLAN\alpha\beta LAL\beta MN\alpha \in T^*$
$CLAL\beta MN\alpha MLLAL\beta MN\alpha \in T^*$
$CLLAN\alpha\beta MLLAL\beta MN\alpha \in T^*$
$CMMLLAN\alpha\beta MMLLAL\beta MN\alpha \in T^*$
$MMMLLAL\beta MN\alpha \in T^*$
$MLLAL\beta MN\alpha \in T^*$
$CLAMN\alpha L\beta ALMN\alpha ML\beta \in T^*$, because $CLApqALpMq \in T$.
$CLLAMN\alpha L\beta LALMN\alpha ML\beta \in T^*$
$CLALMN\alpha ML\beta MLLALMN\alpha ML\beta \in T^*$
$CLLAMN\alpha L\beta MLLALMN\alpha ML\beta \in T^*$
$CMLLAMN\alpha L\beta MMLLALMN\alpha ML\beta \in T^*$
$MMLLALMN\alpha ML\beta \in T^*$
$MMALMN\alpha ML\beta \in T^*$
$CLLMLL\alpha MMML\beta \in T^*$. From the first assumption we have $LLMfamL\alpha \in T^*$.
Then we obtain
$MMML\beta \in T^*$
$ML\beta \in T^*$. Hence, we obtain $L\beta \in M - T^*$.

(R_7): Let $LMM\alpha \in M - T^*$. From this it follows that $MLMM\alpha \in T^*$. But $CLMM\alpha M\alpha \in T^*$, thus $CMLMM\alpha MM\alpha \in T^*$. So, $M\alpha \in T^*$ and $LM\alpha \in T^*$.

Finally $LM\alpha \in M - T^*$.

Theorem 2. $M - T^* = M - S5$.

Proof (outline). In [2] it is proved that $M - S4 = M - S5$. It is enough to show that $M - T^* = M - S4$. Basing on the Theorem 2 in [2] and the Lemma 1, it suffices to prove that the formula $LCLpLLp$ is a thesis of $M - T^*$ system. We shall prove that the formula $MLCLpLLp \in T^*$. It is easy to see that the formula $CCLLMLpMMLLLpMMLCLpLLp$ belong to T.

From the fact that $MMCpLLp$ belongs to T it follows that $MCpLLp \in T^*$. Hence we have:

$CLpMLLp \in T^*$. Furthermore:
$CMLpMMLLp \in T^*$
$CMLLpMMLLLp \in T^*$
$CLpMMLLLp \in T^*$
$CMLpMMMLLLp \in T^*$
$CLLMLpLLMMMLLLp \in T^*$
$CLLMLpLMMMLLLp \in T^*$
$CLMMpMp \in T^*$, because $CLpMLLp \in T^*$.
$CLMMMLLLpMMLLLp \in T^*$
$CLLMLpMMLLLp \in T^*$
$MMLCLpLLp \in T^*$
$MLCLpLLp \in T^*$.

It is easy to verify that the rule (R_4) is provable in the system $M - S4$.

Lemma 2. *For each system $S \in \underline{K}$, if $M - S = M - S4$ then the rule (R_4) is permissible in S.*

Proof. Let $S \in \underline{K}$ and $MM\alpha \in S$. It is true that $MM\alpha \in S$ iff $M\alpha \in M-S$. The assumption reads $M-S = M-S4$, hence $M\alpha \in M-S4$. Furthermore $M\alpha \in M - S4$ iff $\alpha \in MM - S4$. Since $MM - S4 = M - S4$ we get $\alpha \in M - S$, then $M\alpha \in S$.

Theorem 3. *If $S \in \underline{K}$, then $M - S = M - S4$ iff the rule (R_4) is permissible in S.*

The proof of this theorem follows from the Theorem 2 and the Lemma 2.

Summary: From the above remarks it follows that the system T^* is the least one in $\underline{K}$ gaving the property: $M - T^* = M - S4$. It follows that the Perzanowski's system is equal to the system T^*. The axiom $MLCLpLLp$, $MLCMLpLp$ of the Perzanowski's system are provable in the system T^* (see the proof of the Theorem 2).

Theorem 4. *The system T^* and $S4_n$ ($n = 2, 3, \ldots$), where $S4_n$ is a Sobociński's modal system (see [1] page 259), are independent.*

PROOF. It is enough to show that the system $S4_n$ is not contained in the system T^*, because the rule (R_4) is not permissible in the system $S4_n$. Thus, let us take a family of matrices $\{\underline{U}_n\}_{n\geqslant 3}$ given by the following description: $\underline{U}_n = \langle 2^{\underline{R}}; \{\underline{R}\}, \rightarrow, \sim, I_n\rangle$, where $\sim A = \underline{R}\backslash A$; $A \rightarrow B = \sim A \cup B$; $I_n A = \sim C_n \sim A$, where $Cn : 2^{\underline{R}} \rightarrow 2^{\underline{R}}$ and

$$CnA = \begin{cases} A \cup (-\infty, k+1], & \text{when } k = max\{n \in \underline{N}; n \in A\} \text{ and } k < n \\ A \cup (-\infty, n), & \text{when } k = max\{n \in \underline{N}; n \in A\} \text{ and } k \geqslant n \\ A, & \text{when } A \cap \underline{N} = \emptyset \\ \underline{R}, & \text{otherwise.} \end{cases}$$

Here $\underline{N}$ and $\emptyset$ denote: the sets of natural numbers and the empty set, respectively.

It is easy to verify that for any natural number $n \geqslant 2$ $T^* \subset E(\underline{U}_{n+1})$ holds. To show $CL^n pL^{n+1}p \notin E(\underline{U}_{n+1})$ it is enough to substitute the value $(-\infty, 0]$ of the matrix $\underline{U}_{n+1}$ for the variable p. From the proof of the Theorem 4 there follows:

COROLLARY. *The system T^* has infinitely many modalities.*

References

[1] M. J. Creswell, G. E. Hughes, **An Introduction to Modal Logic**, Methuen and Co. Ltd., London 1968.

[2] T. Furmanowski, *Remarks on discussive propositional calculus*, **Bulletin of the Section of Logic** PAN, vol. 4, No. 1 (1975), pp. 33–37.

[3] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, **Studia Societatis Scientiarum Torunensis** 1 (1948), pp. 57–77.

[4] S. Jaśkowski, *O koniunkcji dyskusyjnej w rachunku zdań dla systemów dedukcyjnych sprzecznych*, **Studia Societatis Scientiarum Torunensis** 1 (1949), pp. 171–172.

[5] J. Kotas, *The Axiomatization of S. Jaśkowski's Discussive System*, **Studia Logica** 2 (1974), XXXIII, pp. 195–200.

Institute of Mathematics
Nicholas Copernicus University
Toruń

Chair of Logic
Nicholas Copernicus University
Toruń