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ON LINDENBAUM'S EXTENSIONS (Part B)

An extended version of this abstract will appear in Reports on Mathematical Logic.

Let Cn, be the consequence operation based only on the modus ponens rule (r_0) and substitution rule (r_*).$S^1 = S^c, S^2 = S^{cn}, S^3 = S^{cnka}$ where e.g. S^{cnka} is the set of all well-formed formulas built from propositional variables by means of implication, negation, conjunction and disjunction signs respectively. Tarski has proved in [8], (1934) some theorems concerning the power of the class of Lindenbaum's extensions:

(1) (Tarski): If $\{p \to (q \to p), p \to [(p \to q) \to q], (q \to s) \to [(p \to q) \to (p \to s)]\} = A_1$ and $A_1 \subseteq X \subseteq S^1$ then the only Cn_*-consistent and Cn_*-complete extension of the Cn_*-consistent set X is the class of all two-valued implicational tautologies.

(2) (Tarski): If $\{p \to \sim\sim p, q \to (p \to q), \sim p \to (p \to q), p \to [\sim q \to \sim (p \to q)]\} = A_2$ and $A_2 \subseteq X \subseteq S^2$ then the only Cn_*-consistent and Cn_*-complete extension of the Cn_*-consistent set X is the class of all two-valued tautologies from S^2.

(3) (Tarski): If $\{p \to \sim\sim p, q \to (p \to q), \sim p \to (p \to q), p \to [\sim q \to \sim (p \to q)], p \to (p+q), q \to (p+q), \sim p \to [\sim q \to \sim (p+q)], p \to [q \to (p \cdot q)], \sim p \to \sim (p \cdot q), \sim q \to \sim (p \cdot q)\} = A_3$ and $A_3 \subseteq X \subseteq S^3$ then the only Cn_*-consistent nad Cn_*-complete extension of the Cn_*-consistent set X is the class of all two-valued tautologies from S^3.

Later this Tarski's problem was considered in regard to another systems (cf. [1], [3], [6], [7]). It is fairly obvious that the Lindenbaum's extensions are not uniquely determined (cf. [2], [3], [4], [5], [6]). This paper is a continuation of [1] and is chiefly devoted to the above Tarski's theorems. It concerns the power of these systems which have only one Lindenbaum's extension. Problems considered in this paper were formulated by Professor W. A. Pogorzelski.

We introduce some negations. By S^i we always mean one of the sets: S^1, S^2, S^3. The symbol R_{0^*} denotes the set $\{r_0, r_*\}$. By R_{0^*}-system we mean an ordered pair $\langle R_{0^*}, X\rangle$ ($X \subseteq S^i$ and the rules are over S^i). Let Z^i be the set of all two-valued tautologies from S^i. The symbol T_i denotes the class of all systems $\langle R_{0^*}, X\rangle$ ($X \subseteq S^i$), for which Z^i is the only one Lindenbaum's extension. Systems from T_i will be called "systems with T_i-property".

The following theorems hold:

(4) For any $i \in \{1, 2, 3\}$ there exists R_{0^*}-system with T_i-property weaker than the Tarski's system $\langle R_{0^*}, A_i\rangle$.

(5) Every R_{0^*}-system with T_i-property has non-empty intersection with Tarski's system $\langle R_{0^*}, A_i\rangle$.

Let us note that for some $R \neq R_{0^*}$ (5) isn't true. There are R-systems with T_i-property which have an empty intersection with the system $\langle R_{0^*}, A_i\rangle$. In the present paper we consider only R_{0^*}-systems with T_i-property. Hence and from (4) and (5) we obtain some mutual relations between these systems. We introduce further notations:

$\langle R_{0^*}, X\rangle \overset{+}{+} \langle R_{0^*}, X'\rangle$ iff the sets $Cn_*(X) \cap Cn_*(X')$, $Cn_*(X) - Cn_*(X'), Cn_*(X') - Cn_*(X)$ are non-empty;

$\langle R_{0^*}, X\rangle \prec \langle R_{0^*}, X\rangle$ iff $\langle R_{0^*}, X\rangle$ is a proper subsystem of the system $\langle R_{0^*}, X'\rangle$.

We define now certain class of families of R_{0^*}-systems. It seems not necessary to give an exact definition of this class. We introduce only the symbol $R_I(T_i)$ defined as follows:

a. If $\langle R_{0^*}, X\rangle, \langle R_{0^*}, X'\rangle \in R_I(T_i)$ and $X \neq X'$ then $\langle R_{0^*}, X\rangle \overset{+}{+} \langle R_{0^*}, X'\rangle$

b. If $\langle R_{0^*}, X\rangle \in R_I(T_i)$ and $T \neq A_i$ then $\langle R_{0^*}, X\rangle \overset{+}{+} \langle R_{0^*}, A_i\rangle$.

Of course $R_I(T_i)$ is an element of the above mentioned class. The following theorem holds:

(6) There exists the family of R_{0^*}-systems $R_I(T_i)$ of the power of the continuum.

Note that (6) consists of three theorems for $i = 1, 2, 3$. The second class of the families will consist of families $R_{II}(T_i)$ where $R_{II}(T_i)$ is a family of R_{0^*}-systems (from the language S^i) with T_i-property and satisfies the following two conditions:

a. If $\langle R_{0^*}, X\rangle, \langle R_{0^*}, X'\rangle \in R_{II}(T_i)$ and $X \neq X'$ then $\langle R_{0^*}, X\rangle \overset{+}{+} \langle R_{0^*}, X'\rangle$

b. If $\langle R_{0^*}, X\rangle \in R_{II}(T_i)$ and $X \neq A_i$ then $\langle R_{0^*}, X\rangle \prec \langle R_{0^*}, A_i\rangle$

We can prove a theorem analogous to (6):

(7) There exists the family of R_{0^*}-systems $R_{II}(T_i)$ of the power of the continuum.

The elements of third class of families are families are families $R_{III}(T_i)$. Every $R_{III}(T_i)$ is a set of R_{0^*}-systems (from the language S^i) with T_i-property and satisfies two conditions:

a. If $\langle R_{0^*}, X\rangle, \langle R_{0^*}, X'\rangle \in R_{III}(T_i)$ and $X \neq X'$ then $Cn_*(X) \cap Cn_*(X') = 0$

b. If $\langle R_{0^*}, X\rangle \in R_{III}(T_i)$ and $X \neq A_i$ then $\langle R_{0^*}, X\rangle \overset{+}{+} \langle R_{0^*}, A_i\rangle$.

We have from this definition that

(8) There exists the family of R_{0^*}-systems $R_{III}(T_i)$ which has the cardinality $\aleph_0$.

At last the fourth class of the families consists of families $R_{IV}(T_i)$ which are sets of R_{0^*}-systems (from the language S^i) with T_i-property. $R_{IV}(T_i)$ fulfils the following conditions:

a. If $\langle R_{0^*}, X\rangle$, $\langle R_{0^*}, X'\rangle \in R_{IV}(T_i)$ and $X \neq X'$ then $Cn_*(X) \cap Cn_*(X') = 0$

b. If $\langle R_{0^*}, X\rangle$ $R_{IV}(T_i)$ and $X \neq A_i$ then $\langle R_{0^*}, X\rangle \prec \langle R_{0^*}, A_i\rangle$.

The following theorem can proved:

(9) There exists the family of R_{0^*}-systems $R_{IV}(T_i)$ which has the cardinality $\aleph_0$.

Some of the systems from the above families are not finitely axiomatizable. Now we shall give some examples of axiomatizable systems $\langle R_{0^*}, X_1\rangle$, $\langle R_{0^*}, X_2\rangle$ and $\langle R_{0^*}, X_3\rangle$ $(X_i \subseteq S^i)$ with T_1, T_2, T_3-property respectively, which are proper subsystems of $\langle R_{0^*}, A_i\rangle$. By adding to X_i the axiom φ_1 we obtain examples of systems with T_i-property $\langle R_{0^*}, X_i \cup \{\varphi_1\}\rangle$ such that $\langle R_{0^*}, X_i \cup \{\varphi_1\}\rangle \overset{+}{\underset{+}{}} \langle R_{0^*}, A_i\rangle$.

Example 1.

a. $X_1 = \{p \to p, (p \to q) \to \{[(r \to s) \to p] \to [(r \to s) \to q]\}, p \to [(r \to s) \to p], (r \to s) \to \{[(r \to s) \to q] \to q\}, \{[(r \to s) \to p] \to [(r \to s) \to q]\} \to \{[p \to ((r \to s) \to p)] \to [p \to ((r \to s) \to q)]\}\}$

b. $X_2 = \{\varphi_i \to\sim\sim \varphi_i, \varphi_i \to (\varphi_j \to \varphi_i), \sim \varphi_i \to (\varphi_i \to \varphi_j), \varphi_i \to [\sim \varphi_j \to\sim (\varphi_i \to \varphi_j)]\}_{i=1,2;j=3,4}$ where $\varphi_1 = p \to q, \varphi_2 =\sim p, \varphi_3 = r \to s, \varphi_4 =\sim r$.

c. $X_3 = \{\varphi_i \to\sim\sim \varphi_i, \varphi_i \to [\sim \varphi_j \to (\varphi_i \to \varphi_j)], \varphi_i \to (\varphi_j \to \varphi_i), \sim \varphi_i \to (\varphi_i \to \varphi_j), \varphi_i \to (\varphi_i + \varphi_j), \varphi_j \to (\varphi_i + \varphi_j), \sim \varphi_i \to [\sim \varphi_j \to\sim (\varphi_i + \varphi_j)], \varphi_i \to [\varphi_j \to (\varphi_i * \varphi_j)], \sim \varphi_i \to\sim (\varphi_i * \varphi_j), \sim \varphi_j \to\sim (\varphi_i * \varphi_j) : i \in \{1,2,3,4\}, j \in \{5,6,7,8\}\}$ where $\varphi_1, p \to q, \varphi_2 = p * q$, $\varphi_3 = p + q$, $\varphi_4 =\sim p$, $\varphi_5 = r \to s$, $\varphi_6 = r * s$, $\varphi_7 = r + s$, $\varphi_8 =\sim r$.

Example 2. $\varphi_1 = [p \to (p \to q)] \to (p \to q)$.

It arises a general question whether there exists the minimal R_{0^*}-systems. We didn't solve this problem. We can prove only the following theorem:

(10) a. It doesn't exist the weakest R_{0^*}-system with T_i-property.

b. Form some family of the descending R_{0^*}-systems with T_i-property it doesn't exist the minimal R_{0^*}-system with T_i-property.

Let $\langle R, X\rangle$ be the system where R is set of rules over S^i and $X \subseteq S^i$.

Definition 1. $\langle R', Y\rangle$ is supersystem of the system $\langle R, X\rangle$ directed by $\langle R', X'\rangle$ iff $\langle R, X\rangle \prec \langle R', X'\rangle$ and Y is a Lindenabum extension of the set $Cn(R', X')$. The symbol $\mathfrak{n}$ denotes a cardinal number such that $\mathfrak{n} \leqslant \mathfrak{c}$.

Definition 2. $T_i^{\mathfrak{n}}$ is the class of all R-systems (from S^i) for which there exist such a system $\langle R', X'\rangle$ that the class of supersystems (of the system $\langle R, X\rangle$) directed by $\langle R', X'\rangle$ has the cardinality $\mathfrak{n}$.

The following theorem holds:

(11) There exists $X_0 \subseteq S^i$ such that $\langle R_{0^*}, X_0\rangle \in T_i^{\mathfrak{n}} \cap T_i$, for any $\mathfrak{n} \leqslant \mathfrak{c}$.

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