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STRUCTURAL COMPLETENESS AND THE DISJUNCTION PROPERTY OF INTERMEDIATE LOGICS

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Let At be the set of all propositional variables $p_0, p_1, p_2, \ldots$, let S be the set of well-formed formulas built by means of variables $p_0, p_1, p_2, \ldots$ and connectives: $\rightarrow, *, +, \sim$. By intermediate logic we mean a set $H \subseteq S$ such that $INT \subseteq H \Rightarrow Cn(H) \subseteq L_2$, where INT is the intuitionistic propositional logic, L_2 – two-valued logic and $Cn(H)$ is the smallest set containing H and closed under the substitution rule and the detachment rule.

Kreisel and Putnam [1] have proved that the logic KP obtained by adding to INT the axiom $(p \rightarrow q + r) \rightarrow (\sim p \rightarrow q) + (\sim p \rightarrow r)$ has the following disjunction property:

$$\Phi + \Psi \in KP \Leftrightarrow \Phi \in KP \vee \Psi \in KP.$$

In this paper it is shown that there exist a structural complete intermediate logics with the disjunction property, which was previously conjectured by H. Friedman.

The intermediate logic H is structurally complete ($H \in SCpl$) if and only if for every $\Phi, \Psi \in S$ the following condition holds:

$$\forall_{e:At \rightarrow S}[h^e(\Phi) \in H \Rightarrow h^e(\psi) \in H] \Leftrightarrow (\Phi \rightarrow \Psi) \in H,$$

where h^e is the extension of the function $e : At \rightarrow S$ to the endomorphism of the algebra $\langle S, \rightarrow, *, +, \sim \rangle$, (the notion of structural completeness is introduced by W. A. Pogorzelski in [2]).

Let
$\widetilde{At} = \{\sim p | p \in At\}$;
S = the least set containing At and closed with respect to: $\rightarrow, *, +, \sim$;
$H = \{\Phi \in S | \forall_{e:At\rightarrow\widetilde{S}} h^e(\Phi) \in H\}$, for every intermediate logics with the disjunction property.

We have

THEOREM. *For every intermediate logic H:*

i) $H \in SCpl \Rightarrow KP \subseteq H$

ii) $KP \subseteq H \Rightarrow \widetilde{H} \in SCpl$

iii) $(KP \subseteq H$ *and* $H \in GP) \Rightarrow \widetilde{H} \in GP$.

COROLLARY. $\widetilde{KP} \in SCpl \cap GP$.

References

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[2] W. A. Pogorzelski, *Structural completeness of the propositional calculus*, **Bull. Acad. Polon. Sci.**, Sér. Sci. Math. Astronom. Phys., 19 (1971), pp. 349–351.

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