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CHRISTMAS TREES. ON FREE CYCLIC ALGEBRAS IN SOME VARIETIES OF CLOSURE ALGEBRAS

In [1] J. C. C. McKinsey and A. Tarski using an infinite sequence of distinct closure-algebraic functions of one variable which had been constructed by K. Kuratowski [2], proved that the free cyclic closure algebra (i.e. the free closure algebra with one free generator) is infinite.

The interesting example of an infinite cyclic algebra was found by L. Rieger [3]; however the structure of the free cyclic closure algebra (Riger's problem) is not yet clear.

It is known that there are infinitely many non-equivalent formulas of a single variable in the intuitionistic propositional calculus; in other words, the free cyclic Brouwerian algebra (Rieger – Nishimura's lattice) is infinite.

In the logic of weak law of excluded middle (the system KC) and the "chains" logic (the system LC) the free cyclic KC-algebra (and, hence, LC-algebra) is finite.

This is in contrast with the fact that the set of distinct unary formulas is infinite, as D. Makinson showed [4], in their modal "companions", i.e. in the system $S4.3$ (and, consequently in $S4.2$).

Makinson gave an example of infinite sequence of non-equivalent unary formulas in the modal system $D^* = K3.1$ [5].

However, neither Makinson's nor Kuratowski's sequences contain all the distinct unary formulas.

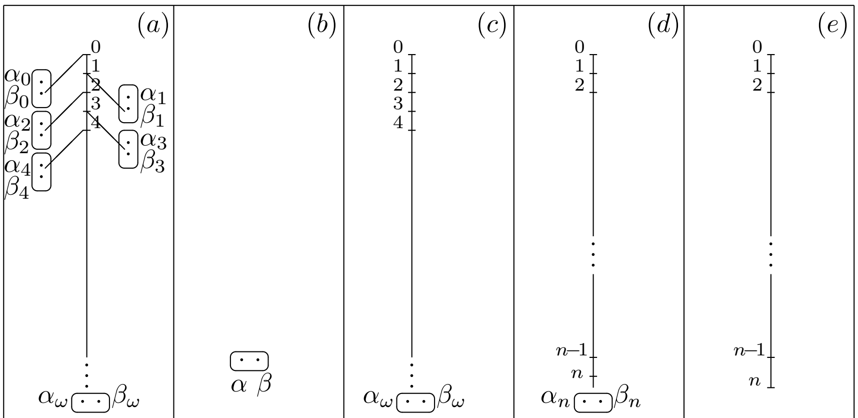
The main purpose of this paper is to give the complete description of free closure algebras in all equational subclasses of the class $S4.3$-algebras mentioned in literature.

Moreover, we give the complete list of the distinct unary formulas of the system D^* and indicate the position of formulas from the sequence of Kuratowski and that of Makinson.

Free Cyclic Algebras Let be $X = \{\omega\} \cup \{\alpha_n, \beta_n\}_{n\in\omega+1}$, where ω is the first infinite ordinal. We define the binary relation R' on the set X in the following way:

(1) $(\forall n, m \in \omega)(nR'm \Leftrightarrow n \geqslant m)$
(2) $(\forall n \in \omega)(\alpha_n R'\alpha_n \;\&\; \beta_n R'\beta_n \;\&\; \alpha_n R'\beta_n \;\&\; \beta_n R'\alpha_n)$
(3) $(\forall n \in \omega)(\alpha_n R'n \;\&\; \beta_n R'n \;\&\; \alpha_\omega R'n \;\&\; \beta_\omega R'n)$

Denote by R the transitive closure of the relation R'. The preordered set (X, R) ("Christmas trees") can be visualised by the following diagram (Fig. 1a)



(X, R) $(X_{\alpha\beta}, R_{\alpha\beta})$ (X_ω, R_ω) (X_n, R_n) $({}^nX, {}^nR)$

Let F' be the field of all finite subsets of all finite subsets of set $X - \{\alpha_\omega, \beta_\omega\}$ and their complements w.r.t. X. Let be $G \in \{E \cup \{\alpha_n\}_{n\in\omega+1}\}$, where $E(\subseteq \omega)$ is the set of all even numbers. Let F be the smallest field, containing the set G and the field F'. It easy to see that elements of the field F are represented in the form $A = (A_1 \cap G) \cap (A_2 \cap (X - G))$. for suitable $A_1, A_2 \in F$.

PROPOSITION 1. *We shall consider the set X as a topological space, the field F being assumed as a basis; then X is the Stone space (o-dimensional compact) and F is identical to the field of all clopen sets of the space X.*

The set $\bigcup\{R(x) : x \in A\}$ (where $R(x) = \{y : xRy\}$), $A \subseteq X$, we denote by $R(A)$.

Analogically, the set $\bigcup\{R^{-1}(x) : x \in A\}$ *(where* $R^{-1}(x) = \{y : yRx\}$*) we denote by* $R^{-1}(A)$.

Proposition 2. *The algebra* $\underline{F} = (F, \cap, \cup, -, R^{-1})$ *is a closure algebra [1], where* $\cup, \cap, -$ *are intersection, union, and complement operation, respectively.*

The following preorder and ordered sets obtained from (X, R) will be useful.
(X_n, R_n) $(n \in \omega + 1)$, where $X_n = R(\{\alpha_n, \beta_n\}) \subseteq X$ and R_n is the restriction of R on X (see, Fig. 1c and 1d); $({}^nX, {}^nR)$ $(n \in \omega)$, where ${}^nX = R(n)$ and ${}^nR$ is the restriction of R on ${}^nX$ (Fig. 1c).

Let F_n (resp., ${}^nF$) be the field of all subsets of $X_n({}^nX)$ for $n \in \omega$; and let be $F_\omega = \{A \cap X_\omega : A \in F\}$.

Proposition 3. $\underline{F} = (F_n, \cup, \cap, -, R_n^{-1})$ *(resp.,* ${}^n\underline{F} = ({}^nF, \cup, \cap, -, {}^nR^{-1})$ *is a finite closure algebra of the equational class of* $S4.3$*-algebras (resp.,* D^**-algebras).*

Proposition 4. *The algebra* $\underline{F}$ *(resp.,* $\underline{F}$*) is isomorphic to a subdirect product of algebras* $\underline{F}_n$ *(resp.,* ${}^n\underline{F}$*),* $n \in \omega$*, and hence* $\underline{F}(\underline{F}_\omega)$ *is* $S4.3$*-algebras (*D^**-algebras).*

Theorem 1. *The algebras* $\underline{F}, \underline{F}_n$ $(n \in \omega + 1)$ *and* ${}^n\underline{F}(n \in \omega)$ *are cyclic; the elements* $G \in \underline{F}$, $G \cap X_n \in \underline{F}_n$ *and* $G \cap^n X \in^n \underline{F}$ *are generators of these algebras respectively. The element* $G \cap^n X(n \in \omega)$, $G \cap X_n$ $(n \in \omega + 1)$ *denote by* ${}^nG$, G_n *respectively. Let be* $X_{\alpha\beta} = \{\alpha, \beta\}$*. Let us define a relation* $R_{\alpha\beta}$ *on* $X_{\alpha\beta}$ *in the following way:* $(\forall x, y \in X_{\alpha\beta}(xR_{\alpha\beta}y)$ *(see, Fig. 1b). (Denote the field of all subsets of* $X_{\alpha\beta}$ *by* $F_{\alpha\beta}$*.*

Proposition 5. *The algebra* $\underline{F}_{\alpha\beta} = (F_{\alpha\beta}, \cup, \cap, -R_{\alpha\beta}^{-1})$ *is cyclic* $S5$*-algebras with the generator* $G_\alpha = \{\alpha\}$.

The Main Theorem. (1) The algebra $\underline{F} \times \underline{F}_{\alpha\beta} \times \underline{F} \simeq \underline{F}_{S4.3}$ is the free cyclic algebra in the $S4.3$-class, where the sign "$\times$" denotes the direct-product operation; the element $(G, G_\alpha, X - G) \in \underline{F}_{S4.3}$ is free generator.

(2) The free cyclic algebras $\underline{F}_{D^*}$ of the variety of D^*-algebras is isomorphic to the algebras $\underline{F}_\omega \times \underline{F}_\omega$; with the free generator $(G_\omega, X_\omega, -G_\omega)$.

In the following items we indicate free cyclic algebras and their generators in the corresponding classes.

(3) $\underline{F}_D \simeq \underline{F}_\omega \times \underline{F}_{\alpha\beta} \times \underline{F}_\omega$; $(G_\omega, G_\alpha, X_\omega - G_\omega)$
(4) $\underline{F}_{K3} \simeq \underline{F} \times \underline{F}$; $(G, X - G)$
(5) $\underline{F}_{K3.2} \simeq \underline{F}_0 \times \underline{F}_0$; $(G_0, X - G)$
(6) $\underline{F}_{S5} \simeq \underline{F}^0 \times \underline{F}_{\alpha\beta} \times \underline{F}^0$; $({}^0G, G_\omega, {}^0X -{}^0 G)$
(7) $\underline{F}_{K2.1} \simeq \underline{F}_{D^*}$
(8) $\underline{F}_{S4.3} \simeq \underline{F}_{S.4.B}$

Remarks to the Main Theorem. Free algebras with n-free generators ($n \in \omega$) in the $S5$-variety, i.e. free monadic algebras [6], wre described in [7] and the item (6), is included only for completeness.

Say that subclass K' of the class K of the algebras is prime with respect to K (K-prime) if there are a family Σ of one variable equations, such that $K' = \{A \in K : \Sigma \models A\}$.

It is known that the class of $S4.3$-algebras is not $S4$-prime [8]; R. Bull showed [8] that the least $S4$-prime class, containing $S4.3$-algebras, may be characterized by the equation
$C - Ca \vee -C(-a \wedge Ca) \vee -C(-C(-a \wedge Ca) \wedge Ca) = 1.$

The corresponding system will be denoted by $S4.B$. This makes clear the item (8) of the theorem. One may show that D^* is not $S4$-prime and $K2.1$ is the least $S4$-prime class, containing D^* (see the item (7)).

Unary formulas in the system D^*. Let p be a propositional variable; let us define two family of unary formulas $\underline{A} = \{A_i(p)\}_{i\in\omega}$ and $\underline{B} = \{B_i(p)\}_{i\in\omega}$ in the following way:

$A_0(p) = p,\ \ A_1(p) = A'_1(p) = \Box p,$
$A'_i(p) = \neg\Diamond(\Diamond(p\&\neg A'_{i-2}(p))\&\neg p)\&\Diamond\Box p$ (i is odd, $i \geqslant 3$)
$A'_i(p) = \neg\Diamond(p\&\neg A'_{i-1}(p))\&\Diamond\Box p$ (i is even, $i \geqslant 2$)
$A_i(p) = A'_i(p)\&\neg A'_{i-1}(p)$
$B_0(p) = \neg p,\ \ B_1(p) = B'_1(p) = \Box\neg p$
$B'_i(p) = \neg\Diamond(\Diamond(\neg p\&\neg B'_{i-2}(p))\&p)\&\Diamond\Box\neg p$ (i is odd, $i \geqslant 3$)
$B'_i(p) = \neg\Diamond(\neg p\&\neg B'_{i-1}(p))\&\Diamond\Box \sim p$ (i is even, $i \geqslant 2$)
$B_i(p) = B'_i(p)\&\neg B'_{i-1}(p).$

Using the main theorem, it is easy to verify that no two of the formulas from the set $\underline{A} \cup \underline{B}$ are equivalent to each other in D^*.

THEOREM 2. *Any formula α is equivalent (in D^*) to a formula of the form $\bigvee_i \&_j \delta_{i,j}(p)$, where*
$\delta_{ij}(p) \in \{A_k(p), \neg A_k(p), B_k(p), \neg B(p)\}_{k\in\omega}$.

For comparison reasons we will give the sequence of Kuratowski and the sequence of Makinson:

$K_0(p) = p, K_1(p) = p\&\Diamond(\Diamond p\&\neg p), K_i(p) =$
$K_{i-1}(p)\&\Diamond(\Diamond K_{i-1}(p)\&\neg K_{i-1}(p)),\ i \geqslant 2.$
$M_0(p) = p, M_{i+1}(p) = \Diamond M_i(p)\&\neg p\ \ (i \text{ is even}),$
$M_{i+1}(p) = \Diamond M_i(p)\&p\ \ (i \text{ is odd}).$

Formulas of Kuratowski sequence are expressed by formulas of $\underline{A} \cup \underline{B}$ as follows

$$K_0(p) = p = A_0(p),$$

$$K_i(p) = K_{i-1}(p)\&\neg A_{2i-1}(p)\&\neg B_{2i}(p),\ i \geqslant 1.$$

For McKinson's sequence we have:

$$M_0(p) = p = A_0(p), \quad M_1(p) = \neg M_0(p)\&\neg B_1(p),$$

$$M_i(p) = M_{i-2}(p)\&\neg A_{i-1}(p)\&\neg B_i(p), \quad i \geqslant 2.$$

Note that in the intermediate system LC there are only six distinct unary formulas as contrasted with the system D^*.

<u>Prime subvarieties of the class D^*.</u> It is known [9], that the class of Boolean algebras is single nontrivial LC-prime subvariety of the class LC. Contrary to this we have

Theorem 4.

(a) *The class of all nontrivial subvarieties of the class D^* is well ordered (by the inclusion relation $\subseteq$), $D_1 \subseteq D_2 \subseteq \ldots \subseteq D^*$, and the type of this ordering is $\omega + 1$;*
(b) *Every D_n $(n \in \omega)$ is prime with respect to D^*.*
(c) *The formula $\neg K_n(p)$ (if n is even) or*

$$\neg(K_{\frac{n-1}{2}}(p)\&\Diamond K_{\frac{n-1}{2}}(\neg p)) \quad \textit{if } n \textit{ is odd)}$$

axiomatizes logical system corresponding to the variety D_n $(n \in \omega)$.

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