Bulletin of the **S**ection of **L**ogic
Volume 4/3 (1975), pp. 103–106

reedition 2012 [original edition, pp. 103–108]

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MODAL SYSTEMS RELATED TO $S4_n$ OF SOBOCIŃSKI

This is an abstract of the paper presented at the seminar of the Section of Logic, Institute of Mathematics Nicholas Copernicus University, held by Professor Jerzy Kotas, Toruń March 1975. The complete version of the paper will be published in Studia Logica.

This paper consists of two parts: in the first one the systems $M^n - S4_n$, $(n = 1, 2, \ldots)$ are being axiomatized with the method given by J. Kotas in [3], in the second, we construct a countable sequence of the different systems T_n^* $(n = 1, 2, \ldots)$, being the least modal systems with the following properties:

i) $T \subsetneq T_n^* \subsetneq S4_n$, $(n = 1, 2, \ldots)$

ii) $M^n - T_n^* = M^n - S4_n$, $(n = 1, 2, \ldots)$.

The paper generalizes the results given in [1] where we considered just the case when $n = 1$.

The investigation of the logic systems of the form $M^n - S$, where S appears to be any normal modal system, was begun by J. Perzanowski in [4].

In this paper we use the notation of Łukasiewicz. The symbols C, N, L, M mean material implication, negation, necessity and possibility, respectively. Small letters $p, q, r, \ldots$ denote sentential variables, while Greek letters $\alpha, \beta, \gamma, \ldots$ any formulas of the considered modal systems.

We put $L^0\alpha = \alpha$ and $L^{n+1}\alpha = LL^n\alpha$. The abbreviation $M^n\alpha$ is defined in the same way. The symbol $\underline{K}_n$ means the class of all normal modal systems (i.e. closed on the substitution, detachment for material implication, and Godel's rule), intermediate between the system T of Feys-von Wright and the Sobociński system $S4_n$. If $S \in \underline{K}_n$, then the symbol

$M^k - S$ denotes the set of all formulas of the system S, which, when preceded k-times by the sign M, become theses of S. $M^k - S$ we call the M^k-counterparts of S. Throughout this paper r_*, r_0, r_L will denote the substitution, detachment for material implication, and Godel's rule, respectively. The R_N denotes the set $\{r_*, r_0, r_L\}$.

If r denotes any rule, then by $L^n r$ is denoted the rule obtained of the rule r by preceding all its premisses and conclusion by the sign L n-times.

For an arbitrary set R and A, of rules and formulas, we put $L^n R = \{L^n r : r \in R\}$ and $L^n A = \{L^n \alpha : \alpha \in A\}$.

The symbol $Cn(R; A)$ will denote the set of all formulas being the consequences of the set A with respect to the set of rules R. Logic systems will be treated as sets of formulas.

§1. Let A and A_n, $(n \geqslant 1)$ be the following sets of formulas:
$\{CCpqCCqrCpr, CpCNpq, CCNppp, CLpp, CLCpqCLpLq\}$ and
$A \cup \{CL^n pL^{n+1}p\}$, $(n \geqslant 1)$.

Notice that the set of formulas A_{n_*} and rules R_N constitute an axiomatics of Sobociński's systems $S4_n$; furthermore the set of formulas A and rules R_N constitute an axiomatics of the system T of Fays - von Wright.

In further consideration we shall employ the following rules of deduction:

(r_n^L)	:	if	$L^n\alpha$,	then α;
(r_n^M)	:	if	$M^n\alpha$,	then α;
(r_n)	:	if	$M^{n+1}\alpha$,	then $M^n\alpha$.

Exploiting the method of axiomatizing of $M - S5$ presented in [3] could be proved:

Theorem 1. *For any* $n \geqslant 1$ *holds* $M^n - S4_n = Cn(L^n R_N \cup \{r_n^L, r_n^M\}; L^n A)$.

Therefore, the systems $M^n - S4_n$ $(n = 1, 2, \dots)$ are finitely axiomatizable. As it is seen, their axiomatics is strictly connected with the axiomatics of $S4_n$.

§2. At present, we are going to define the foretold family of modal systems and give some basic metalogic properties of systems belonging to this family.

Definition 1. $T_n^* = Cn(R_N \cup \{r_n\}; A)$, $(n = 1, 2, \dots)$.

Directly from the definition of the system T_n^* it follows that $T_n^* \in \underline{K}_n$ and $T \neq T_n^*$.

Remark. Replacing the rule (r_n) by the formula $CL^n pM^n L^{n+1} p$ or $CL^n M^{n+1} pM^n p$ we obtain a system equivalent to T_n^*.

Basing on the method given in [3] once more one can prove

Lemma 1. $M^n - T_n^* = Cn(L^n(R_N \cup \{r_n\}) \cup \{r_n^L, r_n^M\}; L^n A)$, $(n = 1, 2, \ldots)$.

Between M^n - counterparts of the systems $S4_n$ and T_n^* holds strict connection, we are going to express in the following lemma.

Lemma 2. $M^n - T_n^* = M^n - S4_n$, $(n = 1, 2, \ldots)$.

Theorem 2. *For every* $S \in \underline{K}_n$: $M^n - S = M^n - S4_n$ *iff* S *is closed on the* (r_n).

This theorem follows from Lemmas 1, 2 and the following property: if $M^n - S = M^n - S4_n$, then $M^{n+1} - S = M^n - S$.

Summary: From the above remarks it follows that the system T_n^* is the least on in $\underline{K}_n$ having the property $M^n - T_n^* = M^n - S4_n$, $(n = 1, 2, \ldots)$.

Theorem 3. *For any* $n \geqslant 1$*:*

i) $T_{n+1}^* \subsetneq T_n^*$
ii) $T_n^* \subsetneq S4_n$.

Theorem 4. *For any* $n, k \geqslant 1$*: the systems* T_n^* *and* $S4_{n+k}$ *are independent.*

Theorem 5. *The system* T_n^* *has infinitely many modalities, where* $n = 1, 2, \ldots$.

The systems T_n^* (as it has been proved) are the least systems in $\underline{K}_n$, whose M^n-counterparts are equal to the M^n-counterparts of $S4_n$. It arises the question what form are the maximal modal systems in the class of all normal modal systems of just the property. The answer to this question is given below.

Definition 2. $S5_n = Cn(R_N : A_n \cup \{CM^n L^n pL^n p\})$, $(n = 1, 2, \ldots)$.

Lemma 3. $M^n - S5_n = M^n - S4_n$, $(n = 1, 2, \ldots)$.

Theorem 6. *For every normal modal system S holds: if $M^n - S = M^n - S4_n$, then $S \subset S5_n$.*

Notice that replacing the formula $CM^nL^npL^np$ by the rule: if $M^nL^n\alpha$, then $L^n\alpha$ we obtain a system equivalent to $S5_n$.

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