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NOTES ON THE RASIOWA – SIKORSKI LEMMA

This paper aims at formulating a condition necessary and sufficient for the existing of a prime filter preserving enumerable infinite joins and meet in a distributive lattice.

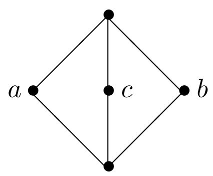
First we prove a simple but very useful

Lemma 1. *Every lattice* $\underline{A} = (A, \cup, \cap)$ *is distributive if and only if* $a \leqslant b$ *provided that there exists* $c \in A$ *such that* $c \neq a$, $c \neq b$ *and* $a \leqslant b \cup c$ *and* $a \cap c \leqslant b$.

Suppose that $\underline{A}$ is a distributive lattice and $a \leqslant b \cup c$, $a \cap c \leqslant b$ for some $c \in A$, $c \neq a$, $c \neq b$. Then

$$a = a \cap a \leqslant a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \leqslant (a \cap b) \cup b = b$$

Other direction, suppose that a lattice A is not distributive and let $\underline{A}$ be the form



It is obvious that $a \leqslant b \cup c$ and $a \cap c \leqslant b$ but $a \leqslant b$ does not hold.

Corollary. *For every lattice* $\underline{A} = (A, \cup, \cap)$ *the next two conditions are equivalent:*

(i) $a \leqslant b$ *if and only if there exists* $c \in A$ *such that* $c \neq a$, $c \neq b$, $a \leqslant b \cup c$ *and* $a \cap c \leqslant b$,

(ii) A *is a distributive lattice.*

It is well known that

LEMMA 2. *In every lattice* A, *if the infinite join and meets concerned exist; then*
$\bigcup_{t\in T}(a_t \cap b) \leqslant b \cap \bigcup_{t\in T} a_t$,
$\bigcap_{t\in T} b_t \cup a \leqslant \bigcap_{t\in T}(b_t \cup a)$.

Let $\underline{A} = (A, \cup, \cap)$ be a lattice and let for every $n \in \omega$, $A_{2n} \subset A$ and $B_{2n+1} \subset A$. We denote

$$a_{2n} = \bigcup_{a\in A_{2n}} a \qquad b_{2n+1} = \bigcap_{b\in B_{2n+1}} b$$

A prime filter ∇ is said to be a Q-filter provided that

(f_1) for every $n \in \omega$ if $a_{2n} \in \nabla$ then $A_{2n} \cap \nabla \neq \emptyset$,
(f_2) for every $n \in \omega$ if $B_{2n+1} \subset \nabla$ then $b_{2n+1} \in \nabla$.

THEOREM. *Let* $\underline{A} = (A, \cup, \cap)$ *be a distributive lattice and let for every* $n \in \omega$, a_{2n} *and* b_{2n+1} *exist. Suppose, that for some* x, y *the inequality* $x \leqslant y$ *does not hold.*

Then there exists a Q-filter ∇ such that $x \in \nabla$ and $y \notin \nabla$ if and only if for every $n \in \omega$, $a', b' \in A$

$(\cap, \cup)$ $\bigcap_{b\in B_{2n+1}}(a' \cup b) \leqslant a' \cup \bigcap_{b\in B_{2n+1}} b$,
$(\bigcup, \cap)$ $b' \cap \bigcup_{a\in A_{2n}} a \leqslant \bigcup_{a\in A_{2n}}(b' \cap a)$.

PROOF. Suppose that there exist $n_0 \in \omega$ such that

$$\bigcap_{b\in B_{2n_0+1}}(a' \cup b) \leqslant a' \cup \bigcap_{b\in B_{2n_0+1}} b$$

does not hold. Then by our assumption there exists a Q-filter ∇ such that $\bigcap_{b\in B_{2n_0+1}}(a' \cup b) \in \nabla$ and $(a' \cup \bigcap_{b\in B_{2n_0+1}} b) \notin \nabla$, for some fixed $n_0 \in \omega$. Hence ∇ is a Q-filter we infer that for every $b \in B_{2n_0+1}, b \in \nabla$ or $a' \in \nabla$ and $a' \notin \nabla$ and there exists $b'_0 \in B_{2n_0+1}$ such that $b'_0 \notin \nabla$. But this is a contradiction.

In a similar way we proof the condition $(\bigcup, \cap)$. If for some $n_0 \in \omega$

$$b' \cap \bigcup_{-a \in A^a_{2n_0}} \leqslant \bigcup_{a \in A_{2n_0}} (b' \cap a)$$

does not hold, then there exists a Q-filter ∇ such that

$$b' \cap \bigcup_{a \in A_{2n_0}} a \in \nabla \text{ and } \bigcup_{a \in A_{2n_0}} (b' \cap a) \notin \nabla.$$

Thus

$$b' \in \nabla \;\&\; \exists_{a \in A_{2n_0}} a \in \nabla \;\&\; \forall_{a \in A_{2n_0}} \sim (b' \in \nabla \;\&\; a \in \nabla)$$

but it is a contradiction which proves necessity.

Other direction. Suppose that for some $x, y, x \leqslant y$ does not hold and the conditions $(\bigcap, \cup)$ and $(\bigcup, \cap)$ are satisfied.

Now we will construct two sequences $(\alpha_n)_{n \in \omega}$ and $(\beta_n)_{n \in \omega}$ of the elements of A such that:

(i) $\alpha_0 = y \qquad \beta_0 = x$,
(ii) $\alpha_{n-1} \leqslant \alpha_n$ and $\beta_{n-1} \geqslant \beta_n$ for $n > 0$,
(iii) $\forall_{n \in \omega} (\beta_{2n+1} \leqslant b_{2n+1} \vee \exists_{b \in B_{2n+1}} b \leqslant \alpha_{n+1})$,
$\forall_{n \in \omega} (\exists_{a \in A_{2n}} \beta_{2n} \leqslant a \vee a_{2n} \leqslant \alpha_{2n})$.
(iv) for every $n \in \omega$ the relation $\beta_n \leqslant \alpha_n$ does not hold.

Suppose that for $k \in \omega$, $\alpha_1, \ldots, \alpha_{2k}$ and $\beta_1, \ldots, \beta_{2k}$ are constructed such that (ii) – (iv) are fulfilled. On account of (iv) we have that the relation $\beta_{2k} \leqslant \alpha_{2k}$ does not hold. By Lemma 1 we have that for every $c \in A$, the relations

$$\beta_{2k} \leqslant \alpha_{2k} \cup c \text{ or } \beta_{2k} \cap c \leqslant \alpha_{2k}$$

do not hold.

Putting $c = b_{2k+1}$ we obtain that $\beta_{2k} \leqslant \alpha_{2k} \cup b_{2k+1}$ does not hold or $\beta_{2k} \cap b_{2k+1} \leqslant \alpha_{2k}$ does not hold.

Consider the first inequality. We have that

$$\sim (\beta_{2k} \leqslant \alpha_{2k} \cup \bigcap_{b \in B_{2k+1}} b).$$

By the condition $(\bigcap, \cup)$ we infer that

$$\sim (\beta_{2k} \leqslant \bigcap_{b \in B_{2k+1}} (\alpha_{2k} \cup b)),$$

i.e.

$$\exists b \in B_{2k+1} \sim (\beta_{2k} \leqslant \alpha_{2k} \cup b).$$

Thus we have

$(*)$ $\exists b \in B_{2k+1} \sim (\beta_{2k} \leqslant \alpha_{2k} \cup b)$ or $\sim (\beta_{2k} \cap b_{2k+1} \leqslant \alpha_{2k})$.

Now if the first condition $(*)$ is satisfied we put

$$\alpha_{2k+1} = \alpha_{2k} \cup b \text{ and } \beta_{2k+1} = \beta_{2k}.$$

If the second condition $(*)$ takes place we put

$$\beta_{2k+1} = \beta_{2k} \cap b_{2k+1} \text{ and } \alpha_{2k+1} = \alpha_{2k}.$$

It is not difficult to check that so defined α_{2k+1} and β_{2k+1} satisfy (ii) – (iv).

Having α_{2k+1} and β_{2k+1} we define the α_{2k+2} and β_{2k+2} in a similar way. By (iv) we have $\beta_{2k+1} \leqslant \alpha_{2k+1}$ does not hold, i.e. that for every $c \in A$, the relations

$$\beta_{2k+1} \leqslant \alpha_{2k+1} \cup c \text{ or } \beta_{2k+1} \cap c \leqslant \alpha_{2k+1}$$

do not hold.

Putting $c = a_{2k+2}$ we have that

$$\sim (\beta_{2k+1} \leqslant \alpha_{2k+1} \cup a_{2k+2}) \text{ or } \exists_{a \in A_{2k+2}} \sim (\beta_{2k+1} \cap a \leqslant \alpha_{2k+1})$$

We define

$$\beta_{2k+2} = \beta_{2k+1} \text{ and } \alpha_{2k+2} = \alpha_{2k+1} \cup a_{2k+2}$$

or

$$\beta_{2k+2} = \beta_{2k+1} \cap a \text{ and } \alpha_{2k+2} = \alpha_{2k+1}$$

In both cases α_{2k+2} and β_{2k+2} satisfied (ii) – (iv).

In this way we defined the sequences $(\alpha_n)_{n \in \omega}$ and $(\beta_n)_{n \in \omega}$. Let I be the ideal generated by the sequence $(\alpha_n)_{n \in \omega}$ and F be the filter generated by the sequence $(\beta_n)_{n \in \omega}$. By (iv) I and F are disjoint and

(v) $\forall_{n \in \omega} (b_{2n+1} \in F \vee \exists_{b \in B_{2n+1}} b \in I)$,

(vi) $\forall_{n \in \omega} (a_{2n} \in I \vee \exists_{a \in A_{2n}} a \in F)$.

It is well known that in a distributive lattice, every filter can be separated from an ideal, disjoint from it, by a prime filter. Let ∇ be a prime filter containing F such that ∇ is disjoint from I. It is obvious that $x \in \nabla$ and $y \notin \nabla$. By (v) and (vi) ∇ is the required Q-filter, which completes the proof of the theorem.

In the same way we can prove a condition necessary and sufficient for the existing of a prime ideal preserving enumerable infinite joins and meets in a distributive lattice.

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