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A PROOF OF A CONJECTURE OF R. SUSZKO

The aim of this paper is to prove that there are no countable matrix adequate for the consequence operation determined by the theorems of $S4$ and detachment rule for the material implication. It follows that Prof. R. Suszko's conjecture from [4], p. 36 is true (although the original conjecture was stated for certain SCI theory it is formally equivalent to a problem solved by our theorem).

The symbol $FOR(S4)$ denotes the set of all formulas built up in usual way from propositional variables and connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, L. By C_{S4} we denote the consequence operation in $FOR(S4)$ determined by the theorems of $S4$ and detachment rule for the material implication. Analogously, the symbol $FOR(INT)$ and C_{INT} denote the set of all formulas of intuitionistic propositional logic (INT) and the consequence operation in $FOR(INT)$ determined by the theorems of INT and detachment rule.

By a C_{S4}-matrix we mean any pair $\langle \mathcal{Z}, F\rangle$ where $\mathcal{Z} = \langle B, \vee, \wedge, \rightarrow, \neg, I\rangle$ is a topological Boolean algebra (see [3]) and F is a filter in a Boolean algebra underlying an algebra $\mathcal{Z}$. The following lemma provides a motivation for this definition:

Lemma 1. *If $\mathcal{M}$ is any matrix such that (i) all the theorems of $S4$ are tautologies of $\mathcal{M}$ and (ii) the detachment rule for the material implication is valid in $\mathcal{M}$ then*

(1) the relation R defined by xRy iff $I(x \leftrightarrow y)$ is distinguished in $\mathcal{M}$ is a matrix congruence in $\mathcal{M}$

(2) the quotient matrix $\mathcal{M}/R$ is a C_{S4}-matrix.

If $\mathcal{M}$ is a C_{S4}-matrix then $C_{\mathcal{M}}$ denotes the matrix consequence determined by $\mathcal{M}$. We say that $\mathcal{M}$ is adequate for C_{S4} iff $C_{\mathcal{M}} = C_{S4}$.

Any pair $\langle \mathcal{A}, F\rangle$ where $\mathcal{A}$ is a pseudo-Boolean algebra and F is a filter in $\mathcal{A}$ is called a C_{INT}-matrix. The symbol $C_{\mathcal{N}}$ denotes the matrix consequence determined by a C_{INT}-matrix $\mathcal{N}$ and $\mathcal{N}$ is said to be adequate for C_{INT} iff $C_{\mathcal{N}} = C_{INT}$.

By T we denote the well-known Tarski-McKinsey transformation ([2], [3]) which maps $FOR(INT)$ into $FOR(S4)$ in the following way:

$Tx = Lx$ for propositional variable x
$T(\alpha \wedge \beta) = T\alpha \wedge T\beta$
$T(\alpha \vee \beta) = T\alpha \vee T\beta$
$T(\alpha \to \beta) = L(T\alpha \to T\beta)$
$T(\neg\alpha) = L\neg T\alpha$ for $\alpha, \beta \in FOR(INT)$.

If $X \subseteq FOR(INT)$ then by TX we denote the set $\{T\alpha : \alpha \in X\}$.

If $\mathcal{M} = \langle\langle B, \wedge, \vee, \to, \neg, I\rangle, F\rangle$ is a C_{S4}-matrix then we define a matrix $\mathcal{M}_L = \langle\langle B_L, \wedge, \vee, \Rightarrow, \urcorner\rangle, F_L\rangle$ putting:

$B_L = \{Ix : x \in B\}$
$x \Rightarrow y = I(x \to y)$
$\urcorner x = I(\neg x)$
$F_L = F \cap B_L$.

Lemma 2 ([1], [3]).

(i) If $\mathcal{M}$ is a C_{S4}-matrix then $\mathcal{M}_L$ is a C_{INT}-matrix.

(ii) Every C_{INT}-matrix is of the form $\mathcal{M}_L$ for some C_{S4}-matrix $\mathcal{M}$.

Lemma 3. *Let $\mathcal{M}$ be a C_{S4}-matrix and let $X \subseteq FOR(INT)$, $\alpha \in FOR(INT)$. Then $\alpha \in C_{\mathcal{M}_L}(X)$ iff $T\alpha \in C_{\mathcal{M}}(TX)$.*

Lemma 4. *Let $\alpha \in FOR(INT), X \subseteq FOR(INT)$. If $T\alpha \in C_{S4}(TX)$ then $\alpha \in C_{INT}(X)$.*

Theorem. *If $\mathcal{M}$ is an adequate matrix for C_{S4} then $\mathcal{M}_L$ is an adequate matrix for C_{INT}.*

Corollary. *Every C_{S4}-matrix adequate for C_{S4} is uncountable.*

To prove the corollary suppose that $\mathcal{M}$ is a countable matrix adequate for C_{S4}. In this case $\mathcal{M}_L$ is by the theorem stated above a countable matrix for C_{INT}. This contradicts Wroński's theorem of [5] which says that there is no countable matrix adequate for C_{INT}.

References

[1] J. C. C. Mc Kinsey, A. Tarski, *On closed element in closure algebras*, **Ann. of Math.** 47 (1946), pp. 122–162.

[2] J. C. C. McKinsey, A. Tarski, *Some theorems about the sentential calculi of Lewis and Heyting*, **J. S. L.** 13 (1948), pp. 1–15.

[3] H. Rasiowa, R. Sikorski, **The mathematics of metamathematics**, PWN, Warszawa 1963.

[4] R. Suszko, **Abolition of the Fregean Axiom**, PAN, Warszawa 1973.

[5] A. Wroński, *On cardinality of matrices strongly adequate for the intuitionistic propositional logic*, **Bulletin of the Section of Logic**, PAN, 3.1 (1974), pp. 34–40.

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