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CUT ELIMINATION IN *SCI*

In [1] the propositional calculus SCI is considered which has the language $\supset, \neg, \equiv$ and can be described by postulates of the classical propositional calculus for $\supset$, $\neg$, plus two schemata: $A \equiv B \supset (A \supset B)$ and $A \equiv B \supset ([C|^{\alpha}_{A}] \equiv C|^{\alpha}_{B}])$, where α is propositional variable, occurring in C, $[C|^{\alpha}_{A}]$ is the result of substituting A for all α in C.

In the present note the Gentzen-type variant of the calculus SCI is constructed and the proof of the normal form (i.e. cutelimination) theorem is outlined for this variant.

The basic results of the paper were announced at the Seminar on mathematical logics and theory of algorithms of the Institute of Applied Mathematics of Tbilisi State University on July 1, 1974 (in 1975 the author got acquainted with [2] where author cut-free variant of the calculus SCI is constructed).

Denote the Gentzen-type variant of the calculus SCI by SCI_c. The axiom schemata of the calculus SCI_c are $\Gamma_1, A, \Gamma_2 \to \Delta_1, A, \Delta_2$, $\Gamma \to \Delta_1, A \equiv A, \Delta_2$. The rules of Gentzen's classical calculus. For $\equiv$ there are four rules. Firstly, these are two $\equiv$-rules, similar to the rules for equality.

$$\frac{[\Gamma|^{\alpha}_{A}], A \equiv B, [\Delta|^{\alpha}_{A}] \to [\Theta|^{\alpha}_{A}]}{[\Gamma|^{\alpha}_{B}], A \equiv B, [\Delta|^{\alpha}_{B}] \to [\Theta|^{\alpha}_{B}]}, \quad \frac{[\Gamma|^{\alpha}_{A}, B \equiv A, [\Delta|^{\alpha}_{A}] \to [\Theta|^{\alpha}_{A}]}{[\Gamma|^{\alpha}_{B}], B \equiv A, [\Delta|^{\alpha}_{B}] \to [\Theta|^{\alpha}_{B}]}, \tag{1}$$

Where α is arbitrary propositional variable occurring in only one number of Γ, Δ, Θ, and this member is different from α.

Secondly, there are two rules, corresponding to the schema $A \equiv B \supset (A \supset B)$.

$$\frac{\Gamma_1, A \supset B, A \equiv B, \Gamma_2 \to \Delta}{\Gamma_1, A \equiv B, \Gamma_2 \to \Delta}, \quad \frac{\Gamma_1, B \supset A, A \equiv B, \Gamma_2 \to \Delta}{\Gamma_1, A \equiv B, \Gamma_2 \to \Delta}. \tag{2}$$

Note that rules (2) can be replaced by the rule

$$\frac{\Gamma_1, A \supset B, B \supset A, A \equiv B, \Gamma_2 \to \Delta}{\Gamma_1, A \equiv B, \Gamma_2 \to \Delta}. \tag{3}$$

It can be proved (with the help of cut rule) that SCI and SCI_c are equivalent. Note that figure (1) without restriction on α is easily simulated by rules (2) and $\supset\to$. Denote by $SCI^{\equiv}$ the calculus obtained from SCI_c by removing all structural rules, including the cut.

The number of occurrences of $\supset, \neg$ in the formula A will be called the degree of A. The propositional variables and the formulas of the form $A \equiv B$ are called $\equiv -$ formulas. The sequent will be called $\equiv$-sequent if all the formulas occurring in this sequent are $\equiv$-formulas.

Denote by $\mathcal{J}_0$ the auxiliary calculus whose postulates are axiom schemata $\Gamma \to \Delta_1, A \equiv A, \Delta_2$ and $\Gamma_1, A, \Gamma_2 \to \Delta_1, A, \Delta_2$ and all $\equiv$-rules.

The notations of a succedent, weeded, right-sided, left-sided deduction introduced in [3] carry over in a natural way to the calculus $\mathcal{J}_0$.

We call an application of a $\equiv$-rule in some deduction regular if only $\equiv$-rules are applied above it. We call a deduction regular if all applications in it of $\equiv$-rules are regular. The following is proved similarly to Lemma 11 of [3].

Lemma 1. *Any deduction of a sequent S in $SCI^{\equiv}$ can be transformed in a regular deduction of S in $SCI^{\equiv}$.*

In order to prove this lemma rules (2) are used in essential way. After replacing the formulas of the form $\imath = s$ of the calculus $K^{=}_{oc}$ [3] by the formulas of the form $A \equiv B$ and arbitrary formulas of the calculus $K^{=}_{oc}$ by the formulas of the calculus SCI Lemma 1 of [3] becomes true for SCI_c. The role of axioms over elementary formulas is played by $\equiv$-sequents. Lemma 2, 3, 4, 5 of [3] are completely retained for $\mathcal{J}_0$.

Hence applications of $\equiv$-rules can be restricted to succedent ones with premises and conclusions being $\equiv$-sequents.

Lemma 2. *Any deduction of $\Gamma_1, A \equiv A, \Gamma_2 \to \Delta$ in $\mathcal{J}_0$ can be transformed in a deduction of $\Gamma_1, \Gamma_2 \to \Delta$ in $\mathcal{J}_0$.*

This lemma is proved similarly to Lemma 7 of [3].

Lemma 3. *In $\mathcal{J}_0$ the cut is admissible with respect to $\equiv$-formulas.*

PROOF. Only three cases are possible.

CASE 1. The cut formula has the form $A \equiv A$. Then Lemma 2 is used.

CASE 2. The cut formula is the propositional variable. Consider the weeded deductions of cut premises. In view of our restrictions on $\equiv$-rules both cut premises are axioms.

CASE 3. The cut formula has the form $A \equiv B$ and the cut in question is

$$\frac{\Gamma_1, \Gamma_2 \to \Delta_1, A \equiv B, \Delta_2;\ \Gamma_1, A \equiv B, \Gamma_2 \to \Delta_1, \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}.$$

In view of Lemmas 3, 4, 5 for $\mathcal{J}_0$ the deduction of the left premise of a cut can be written in the form

$$\frac{\dfrac{\Gamma_1, \Gamma_2 \to \Delta_1, A \equiv A, \Delta_2}{}}{\dfrac{\cdots}{\Gamma_1, \Gamma_2 \to \Delta_1, A \equiv B, \Delta_2}}.$$

Denote the deduction of a sequent $\Gamma_1, A \equiv B, \Gamma_2 \to \Delta_1, \Delta_2$ in $\mathcal{J}_0$ by $\mathcal{D}$. Consider the deduction

$$\frac{\mathcal{D}\left\{\dfrac{\cdots}{\Gamma_1, A \equiv B, \Gamma_2 \to \Delta_1, \Delta_2}\right.}{\dfrac{\cdots}{\Gamma_1, A \equiv A, \Gamma_2 \to \Delta_1, \Delta_2}}$$

and apply Lemma 2 to the bottom.

THEOREM. *Cut is admissible in* $SCI^{\equiv}$.

This theorem is proved according to the usual Gentzen schema applying Lemmas 2 and 3 in the basis.

Note that the described method for proving the cut elimination in SCI_c using Lemma 6 of [3] provides a new schema of cut elimination from the predicate calculus with the equality $K^{=}_{oc}$ [3].

Define the length of the formula A as the number of all occurrences of $\supset, \neg, \equiv$ in A. The length of the formula A will be denoted by A^{ℓ}. The application of $\equiv$-rule is said to be nonprolonging if the condition $A^{\ell} \leqslant B^{\ell}$ is satisfied. In deduction in $\mathcal{J}_0$ the applications of $\equiv$-rules can be restricted to be nonprolonging. This gives the new proof of the decidability of the calculi $\mathcal{J}_0$ and $SCI^{\equiv}$ which is more efficient than one in [1].

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References

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