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MODAL SYSTEMS PLACED IN THE "TRIANGLE" $S4 - T_1^* - T$

This is an abstract of the paper which will be published in Studia Logica. This paper was presented at the seminar of the Section of Logic, Institute of Mathematics, Nicholas Copernicus University held by Professor Jerzy Kotas in Toruń in March 1975.

In this paper we are going to define a certain family of modal systems. The systems of this family will be denoted by T_n^k ($n \geqslant 1$, $1 \leqslant k \leqslant n$). The subject of investigations are metalogic connections between the systems: T_n^k, $S4$ (Lewis' system), T (Feys' system), $S4_n$ ($n \geqslant 2$, Sobociński's system – see [2] page 259), T_n^* ($n \geqslant 1$, the systems considered in [1]).

We shall use the notations of Łukasiewicz. The symbols C, N, L, M mean material implication, negation, necessity and possibility respectively. Small letters $p, q, r, \ldots$ will denote sentential variables, while Greek letters $\alpha, \beta, \gamma, \ldots$ will denote any formulas of the considered modal systems. We put $L^0\alpha = \alpha$, $L^{k+1}\alpha = LL^k\alpha$ and $M^0\alpha = \alpha$, $M^{k+1}\alpha = MM^k\alpha$; where k denotes any natural number. If S is any modal system and k is any natural number, then by an M^k – counterpart of the system S we call the set of such formulas of S which, preceded by the sign M (k-times), become theses of S. It will be denoted by $M^k - S$.

Let $\underline{A}$ be a set of the formulas of any system S and $(R_1), \ldots, (R_n)$ be arbitrary deduction rules. The symbol $Cn(\underline{A}; R_1, \ldots, R_n)$ denotes the set of all formulas which are consequences of the set $\underline{A}$ with respect to the deduction rules $(R_1), \ldots, (R_n)$.
The considered systems will be treated as sets of formulas.

Let $\underline{A}_n$ be the set consisting of the following formulas:

$(A_1):$ $CCpqCCqrCpr$;
$(A_2):$ $CpCNpq$;
$(A_3):$ $CCNppp$;
$(A_4):$ $CLpp$;
$(A_5):$ $CLCpqCLpLq$;
$(A_6^n):$ $CL^npL^{n+1}p$, $(n = 1, 2, 3, \ldots)$.

In further considerations we are going to employ the following deduction rules:

(R_1) : substitution rule;
(R_2) : if α and $C\alpha\beta$, then β;
(R_3) : if α, then $L\alpha$;
(R_4^k) : if $M^{k+1}\alpha$, then $M^k\alpha$, $(k = 1, 2, 3, \ldots)$.

There will be defined a certain family of modal systems in the following way:

Definition 1. $T_n^k = Cn(\underline{A}_n; R_1, R_2, R_3, R_4^k)$, where $n \geqslant 1$ and $1 \leqslant k \leqslant n$.

Notice that for $n \geqslant 1$ the system T_n^k is equal to $S4_n$; where by $S4_1$ we mean the Lewis system $S4$.

Remark. Replacing the rules (R_4^k) be the formula $CL^kpM^kL^{k+1}p$ or by the formula $CL^kM^{k+1}pM^kp$ we obtain a modal system deductively equivalent to T_n^k $(n \geqslant 1,\ 1 \leqslant k \leqslant n)$.

In [1] it is shown, that the systems T_n^* and $S4_{n+k}$ $(n = 1, 2, 3, \ldots$ and $k = 1, 2, 3, \ldots)$ are independent and moreover, that the system T_n^* $(n = 1, 2, 3, \ldots)$ has infinitely many modalities.

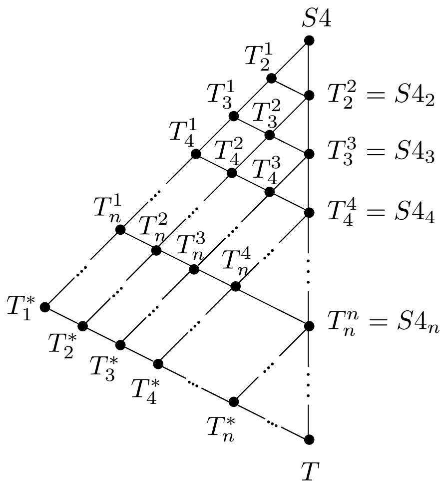
One the base of the above remarks one can prove the following intuitively clear

Theorem 1. *The following properties hold:*

i) $T_n^k \subsetneq T_n^{k-1}$ *for* $n \geqslant 2$ *and* $2 \leqslant k \leqslant n$
ii) $T_{n+1}^k \subsetneq T_n^k$ *for* $n \geqslant 1$ *and* $1 \leqslant k \leqslant n$
iii) $T_k^* \subsetneq T_n^k$ *for* $k \geqslant 1$ *and* $n \geqslant k$.

Basing on the Theorem 1 the mutual plight of the systems T_n^k, $S4$, T,

$S4_n$, T_n^*, may be described as on the diagram



where the systems placed in the *lower* part of the diagram are contained properly in the systems placed *higher* in it. As one can see, for any system T_n^* $(n \geqslant 1)$ it is easy to define infinitely many modal systems finitely axiomatizable, intermediate between T_n^* and $S4_n$.

Directly from this theorem and Theorem 1 (see [1]) follows

COROLLARY. *For any natural numbers $k \geqslant 1$ and $n \geqslant k$ holds:*

$$M^k - T_k^* = M^k - T_n^k = M^k - S4_k.$$

The systems $S4_n$ have, as it was shown in the paper [1], the following property: for each system $S4_n$ there exists a modal system different from $S4_n$ and T, the least one in the class of all normal modal systems which are intermediate between T and $S4_n$ such, that their M^n-counterpart is equal to the $M^n - S4_n$. It is well-known, that the systems adequate to Sobociński's systems $S4_n$, among the intermediate systems between Brouwerian system B and Lewis system $S5$ are the systems of Thomas T_n^+ (see [2] page 260).

Then, in the natural way the question appears, whether for any system T_n^+ there exists (similarly as for the system $S4_n$) a system different from T_n^+ and B, the least one in the class of all normal modal systems which are intermediate between B and T_n^+ such, that their M^n-counterpart is equal to $M^n - T_n^+$.

The answer to this question is given by the following theorem:

THEOREM 2. *For any natural number $n \geqslant 1$ and any modal system S, intermediate between B and T_n^+ closed on the substitution and detachment rules holds:*

$$M^n - S = M^n - T_n^+ \text{ iff } S = T_n^+.$$

From the Theorem 2 it follows, that there is no such modal system even in the class of all modal systems closed on the substitution and detachment rules which are intermediate between the system B and T_n^+.

In this paper and in [1] there has been defined a family of modal systems intermediate between the system T and $S4_n$. Therefore, the question arises, if in a similar way one can define a family of modal systems intermediate between B and T_n^+.

The answer to this question is given by the following theorem:

THEOREM 3. *For any natural number $n \geqslant 1$ the system obtained by closing the Brouwerian system B on the rule (R_4^n) is deductively equivalent to the system T_n^+, where we assume that the denotion T_1^+ means the same as $S5$.*

Notice that extending the Brouwerian system B by the formula $CM^{n+1}pM^np$ we obtain the modal system deductively equivalent to Thomas' modal system T_n^+.

Then, the above theorem provides some non-trivial information.

References

[1] J. J. Błaszczuk, W. Dziobiak, *Modal Systems related to $S4_n$ of Sobociński*, forthcoming in **Bulletin of the Section of Logic**, PAN, Vol. 4 (1975), no. 3.

[2] G. H. Hughes, M. J. Creswell, **An Introduction to Modal Logic**, London, 1968.

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