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ON NORMAL AGASSIZ SYSTEMS OF ALGEBRAS

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The concept of Agassiz system of algebras and Agassiz sum was introduced by G. Grätzer and J. Sichler in [3]. In this paper we intend to discuss a modification of the concepts above which seems to be more advantageous.

Our notation and nomenclature is basically that of Grätzer [1] but we prefer to have the symbol "=" reserved exclusively for the "real" equality and thus, by identities of a given type τ we mean expressions of the form $\mathbf{p} \equiv \mathbf{q}$ where $\mathbf{p}$ and $\mathbf{q}$ are polynomial symbols built up in the usual way (see [1]) be means of some variables from the list $\mathbf{x}_1, \mathbf{x}_2, \ldots$ and some operation symbols of type τ. The set of such polynomial symbols will be denoted by $\mathbf{P}(\tau)$ and they will be referred to as polynomial symbols of type τ.

A mapping $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ will be called a naming functor (comp. [3]) if for every $\mathbf{p} \in \mathbf{P}(\tau)$ the variables of $\mathbf{p}$ and $N(\mathbf{p})$ are the same. Let us note that to assure the existence of a naming functor from $\mathbf{P}(\tau)$ into $\mathbf{P}(\varrho)$ it is necessary and sufficient to require that the range of type ϱ contains 0 whenever the range of τ does and the range of ϱ contains a number greater than 1 whenever the range of τ does.

Given a naming functor $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$, we say that an algebra $\mathbf{B}$ of type ϱ belongs to the structurality class of N ($\mathbf{B} \in SC(N)$) if and only if for every n-ary ($n \geqslant 1$) polynomial symbol $\mathbf{p} \in \mathbf{P}(\tau)$ and every $\mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathbf{P}(\tau)$ the following conditions hold:

(i) $N(\mathbf{p}(\mathbf{q}_1, \ldots, \mathbf{q}_n)) \equiv N(\mathbf{p})(N(\mathbf{q}_1), \ldots, N(\mathbf{q}_n)) \in Id(\mathbf{B})$,
(ii) $N(\mathbf{p}) \equiv \mathbf{p} \in Id(\mathbf{B})$ whenever $\mathbf{p}$ is a variable.

Observe that $SC(N)$ always is an equational class of algebras.

An identity $\mathbf{p} \equiv \mathbf{q}$ of type τ is called N-regular in a class of algebras I of type ϱ (comp. [3]) if $N(\mathbf{p}) \equiv N(\mathbf{q}) \in Id(I)$. Let the symbol $Id_N(I)$

denotes the set of all N-regular identities in I. It is easy to see that $Id_N(I)$ is a closed set of identities whenever $I \subseteq SC(N)$.

An identity $\mathbf{p} \equiv \mathbf{q}$ will be called symmetric if $\mathbf{p} = \mathbf{q}$ or each of $\mathbf{p}, \mathbf{q}$ is not variable. If Σ is a set of identities then by $Sm(\Sigma)$ we denote the set of all symmetric identities from Σ. Observe that $Sm(\Sigma)$ is a closed set of identities whenever Σ does.

An identity will be called inconsistent if it holds only in degenerate (one-element) algebras. It is easy to see that an asymmetric identity having no variable occurring jointly on both its sides must be inconsistent.

Let us define a normal Agassiz system of algebras as a quadruplet $\mathbf{S} = (\mathbf{B}, (\mathbf{A}_b : b \in B), (h_{bc} : \langle b, c\rangle \in R), N)$ such that:

(i) $\mathbf{B}$ is an algebra of a type ϱ (indexing algebra of $\mathbf{S}$),
(ii) $(\mathbf{A}_b : b \in B)$ is a family of algebras of a type τ (indexed family of $\mathbf{S}$) such that $A_b \cap A_c = \emptyset$ whenever $b, c \in B, b \neq c$,
(iii) $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ is a naming functor (naming functor of $\mathbf{S}$) such that $\mathbf{B} \in SC(N)$,
(iv) $R \subseteq B \times B$ is a transitive relation such that for every n-ary ($n \geqslant 1$) operation symbol $\mathbf{f}$ of type τ if $b_1, \ldots, b_n \in B$ and $b = (N(\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)))_{\mathbf{B}}$ $(b_1, \ldots, b_n)$ then $\langle b_i, b\rangle \in R$, $i = 1, \ldots, n$,
(v) $(h_{bc} : \langle b, c\rangle \in R)$ is a family of homomorphisms (h_{bc} is a homomorphism of $\mathbf{A}_b$ into $\mathbf{A}_c$) such that $h_{cd} \circ h_{bc} = h_{bd}$ whenever $\langle b, c\rangle, \langle c, d\rangle \in R$.

The sum of the normal Agassiz system $\mathbf{S}$ is an algebra $\mathbf{A}$ of type τ with the base set $A = \bigcup(A_b : b \in B)$ and the basic operations defined as follows:

(i) if $\mathbf{f}$ is n-ary ($n \geqslant 1$) operation symbol of type τ, $a_1, \ldots, a_n \in A$, $a_i \in A_{b_i}$, $i = 1, \ldots, n$ and $b = (N(\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)))_{\mathbf{B}}(b_1, \ldots, b_n)$ then $(\mathbf{f})_{\mathbf{A}}(a_1, \ldots, a_n) = (\mathbf{f})_{\mathbf{A}_b}(h_{b_1 b}(a_1), \ldots, h_{b_n b}(a_n))$,
(ii) if $\mathbf{f}$ is a nullary operation symbol of type τ then $(\mathbf{f})_{\mathbf{A}} = (\mathbf{f})_{\mathbf{A}_c}$ where $c = (N(\mathbf{f}))_{\mathbf{B}}$.

From now on it will be always tacitly assumed that K, I are nonempty classes of algebras of types τ, ϱ respectively and $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ is a naming functor such that $I \subseteq SC(N)$. We will use the notation (I, K, N) for the class of all normal Agassiz systems whose naming functor is N, indexing algebras are of the class I and indexed families consist of isomorphic copies of algebras of K. The symbol $lim(I, K, N)$ will be used to denote the class of all isomorphic copies of sums of normal Agassiz systems of (I, K, N).

We are not sure that our understanding of the definition of Agassiz system and Agassiz sum as stated in [3] is in accordance with the authors' intention but nevertheless, we have an evidence that the concept of [3] do not coincide with ours because we have only the following weakened version of the theorem of [3]:

PROPOSITION 1. $Sm(Id_N(I) \cap Id(K)) \subseteq Id(lim(I, K, N)) \subseteq Id_N(I) \cap Id(K)$.

The reader will find no difficulty in proving the proposition above with the help of the following:

LEMMA. *Let* $\mathbf{A}$ *be the sum of a normal Agassiz system* $\mathbf{S} = (\mathbf{B}, (\mathbf{A}_b : b \in B), (h_{bc} : \langle b, c\rangle \in R), N)$. *Let* $\mathbf{p}$ *be n-ary* ($n \geqslant 1$) *polynomial symbol of the type of* $\mathbf{A}$. *Let* $a_1, \ldots, a_n \in A$, $a_i \in A_{b_i}$, $i = 1, \ldots, n$ *and* $(\mathbf{p})_{\mathbf{A}}(a_1, \ldots, a_n) \in A_b$. *Then the following conditions are satisfied:*

- *(i)* $b = (N(\mathbf{p}))_{\mathbf{B}}(b_1, \ldots, b_n)$,
- *(ii) if* $\langle b, c\rangle \in R$ *and* $\mathbf{x}_i$ *is a variable of* $\mathbf{p}$ *then* $\langle b_i, c\rangle \in R$,
- *(iii) if* $\mathbf{p}$ *is not a variable and* $\mathbf{x}_i$ *is a variable of* $\mathbf{p}$ *then* $\langle b_i, b\rangle \in R$,
- *(iv) if* $\langle b, c\rangle, \langle b_1, c\rangle, \ldots, \langle b_n, c\rangle \in R$ *then* $h_{bc}((\mathbf{p})_{\mathbf{A}}(a_1, \ldots, a_n)) = (\mathbf{p}_{\mathbf{A}_c}(h_{b_1c}(a_1), \ldots, h_{b_nc}(a_n))$,
- *(v) if* $\mathbf{p}$ *is not a variable then* $(\mathbf{p}_{\mathbf{A}}(a_1, \ldots, a_n) = (\mathbf{p})_{\mathbf{A}_b}(a_1^*, \ldots, a_n^*)$ *where* $a_i^* = h_{b_ib}(a_i)$ *whenever* $\mathbf{x}_i$ *is a variable of* $\mathbf{p}$ *and* a_i^* *is an arbitrary element of* A_b *otherwise.*

Following Grätzer [2] by a restriction of an algebra $\mathbf{A}$ we mean an endomorphism $g : \mathbf{A} \to \mathbf{A}$ such that $g \circ g = g$ A retraction g is non-trivial if $g(a) \neq a$ for some $a \in A$. In order to get a characterization of identies holding in $lim(I, K, N)$ let us note the following sharpened version of the Proposition 1:

PROPOSITION 2.

- *(i) If algebras of* K *have only trivial retractions then* $Id(lim(I, K, N)) = Id_N(I) \cap Id(K)$.
- *(ii) If an algebra of* K *has a non-trivial retraction then* $Id(lim(I, K, N)) = Sm(Id_N(I) \cap Id(K))$.

Let us agree to use the symbol T for trivial naming functor such that $T(\mathbf{p}) = \mathbf{p}$ for every polynomial symbol $\mathbf{p}$ of a considered type. By Proposition 1 and 2 (ii) we get the following:

COROLLARY. *An equational class K containing a non-degenerate algebra can be defined by means of symmetric identities if and only if $lim(K, K, T) \subseteq K$.*

To get a characterization of $Id_N(I) \cap Id(K)$ in terms of normal Agassiz systems we need a refinement of the class (I, K, N). Let us say that a normal Agassiz system $\mathbf{S} = (\mathbf{B}, (\mathbf{A}_b : b \in B), (h_{bc} : \langle b, c\rangle \in R), N)$ is fine if it satisfies the following condition:

(F) for every $b \in B$ there exists $c \in B$ such that $\langle b, c\rangle \in R$ and h_{bc} is an embedding.

Denoting by $(I, K, N)_F$ the class of all fine Agassiz systems from (I, K, N) we can state the following:

PROPOSITION 3. *$Id(lim(I, K, N)_F) = Id_N(I) \cap Id(K)$.*

It is easy to see that fine Agassiz systems of algebras can be viewed as a generalization of direct systems considered by J. Płonka in [4] and Proposition 3 as a generalization of the Theorem 1 of [4] (comp. [3]).

Investigating classes of algebras of the kind $lim(I, K, N)$ from algebraic point of view we can confine ourselves to the cases when K and I are of the same type and the naming functor N is trivial. To be more exact we have the following:

PROPOSITION 4. *There exists a class of algebras I_N of the same type as K such that $lim(I, K, N) = lim(I_N, K, T)$ and $Id_N(I) = Id(I_N)$.*

The proposition above shows in fact, that naming functors can be eliminated and suggests the following definition.

A triplet $\mathbf{S} = (\mathbf{B}, (\mathbf{A}_b : b \in B), (h_{bc} : \langle b, c\rangle \in R))$ is a normal system of algebras if $\mathbf{S}' = (\mathbf{B}, (\mathbf{A}_b : b \in B), (h_{bc} : \langle b, c\rangle \in R), T)$ is a normal Agassiz system of algebras. The sum of the normal system $\mathbf{S}$ is defined as that of $\mathbf{S}'$. For I, K being non-empty classes of algebras of the same type we write $lim(I, K)$, $lim(I, K)_F$ instead of $lim(I, K, T)$, $lim(I, K, T)_F$ respectively. An immediate consequence of the facts stated previously is the following:

PROPOSITION 5.

(i) *If algebras of K have only trivial retractions then* $Id(lim(I,K)) = Id(I \cup K)$.

(ii) *If an algebra of K has a non-trivial retraction then* $Id(lim(I,K)) = Sm(Id(I \cup K))$.

(iii) $Id(lim(I,K)_F) = Id(I \cup K)$.

(iv) *An identity is preserved under the formation of sums of normal systems of algebras if and only if it is symmetric or inconsistent.*

References

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