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ON WEAK AGASSIZ SYSTEMS OF ALGEBRAS

This is an abstract of the paper submitted to Colloqium Mathematicum.

In [2] we dealt with the concept of normal Agassiz system of algebras being a modification of Agassiz system introduced by G. Grätzer and J. Sichler in [1]. Now we want to discuss a generalization of the concept of normal Agassiz system in the direction suggested by the definition of naming functor given by Grätzer and Sichler in [1]. The present paper is a continuation of [2] and thus all notational and terminological conventions of [2] will be used freely. For the sake of convenience in some situation we shall use the notation $\mathbf{p} \equiv_K \mathbf{q}$ interchangeably with $\mathbf{p} \equiv \mathbf{q} \in Id(K)$. In several cases, to make evident the length of a sequence of polynomial symbols we shall use the notation $(\mathbf{p}, \ldots, \mathbf{p})^k$ for k-termed ($k \geqslant 1$) sequence of $p - s$.

Given a naming functor $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ (see [2]), we say that an algebra $\mathbf{B}$ of type ϱ belongs to the weak structurality class of N ($\mathbf{B} \in WSC(N)$) if and only if for every n-ary ($n \geqslant 1$) polynomial symbol $\mathbf{p} \in \mathbf{P}(\tau)$ and every $\mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathbf{P}(\tau)$,

$$N(\mathbf{p}(\mathbf{q}_1, \ldots, \mathbf{q}_n)) \equiv N(\mathbf{q})(N(\mathbf{q}_1), \ldots, N(\mathbf{q}_n)) \in Id(\mathbf{B}).$$

Obviously $WSC(N)$ is always an equational class and $SC(N) \subseteq WSC(N)$. Note also that in order to assure the closedness of the set of identities $Id_N(I)$ it is sufficient to assume that $I \subseteq WSC(N)$ (comp. [2]). Although the class $WSC(N)$ is defined simply by dropping the condition (ii) of [2], we have the following:

Lemma 1. *If $I \subseteq WSC(N)$ and there exists a variable $\mathbf{x}_k$ such that $N(\mathbf{x}_k) \equiv \mathbf{x}_k \in Id(I)$ then $I \subseteq SC(N)$.*

A naming functor $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ will be called standard if for every n-ary ($n \geqslant 1$) polynomial symbol $\mathbf{p} \in \mathbf{P}(\tau)$ and every $\mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathbf{P}(\tau)$, $N(\mathbf{p}(\mathbf{q}_1, \ldots, \mathbf{q}_n)) = N(\mathbf{p})(N(\mathbf{q}_1), \ldots, N(\mathbf{q}_n))$.

Lemma 2. *If a naming functor N is standard then $SC(N)$ is the class of all algebras of type ϱ.*

With each naming functor $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ we associate a mapping $\overline{N} : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ defined as follows:

(i) $\overline{N}(\mathbf{x}_i) = \mathbf{x}_i$ for every variable $\mathbf{x}_i$,
(ii) $\overline{N}(\mathbf{f}) = N(\mathbf{f})$ if $\mathbf{f}$ is a nullary operation symbol of type τ,
(iii) $\overline{N}(\mathbf{f}(\mathbf{p}_1, \ldots, \mathbf{p}_n)) = N(\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n))(\overline{N}(\mathbf{p}_1), \ldots, \overline{N}(\mathbf{p}_n))$ if $\mathbf{f}$ is n-ary ($n \geqslant 1$) operation symbol of type τ and $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbf{P}(\tau)$.

Lemma 3. *If $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ is a naming functor then the following conditions hold:*

(i) $\overline{N}$ is a standard naming functor,
(ii) if $I \subseteq WSC(N)$ then $N(\mathbf{p}) \equiv_I \overline{N}(\mathbf{p}(N(\mathbf{x}_1), \ldots, N(\mathbf{x}_n))$ for every n-ary ($n \geqslant 1$) polynomial symbol $\mathbf{p} \in \mathbf{P}(\tau)$,
(iii) if $I \subseteq SC(N)$ then $N(\mathbf{p}) \equiv_I \overline{N}(\mathbf{p})$ for every $\mathbf{p} \in \mathbf{P}(\tau)$,
(iv) $N(\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)) = \overline{N}(\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n))$ for every n-ary ($n \geqslant 1$) operation symbol $\mathbf{f}$ of type τ,
(v) if $I \subseteq WSC(N)$, $\mathbf{x}_k$ is a variable then for every polynomial symbol $\mathbf{p} \in \mathbf{P}(\tau)$ being not a variable, $N(\mathbf{x}_k)(\overline{N}(\mathbf{p}), \ldots, \overline{N}(\mathbf{p}))^k \equiv_I \overline{N}(\mathbf{p})$.

Let us define a weak Agassiz system of algebras by replacing the condition (iii) of the definition of normal Agassiz systems (see [2]) with the following:

(iii) $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ is a naming functor such that $\mathbf{B} \in WSC(N)$.

The sum of a weak Agassiz system will be defined exactly as that of a normal Agassiz system and the notations $[I, K, N]$, $lim[I, K, N]$ will be used for weak Agassiz systems in the same manner as (I, K, N), $lim(I, K, N)$ for normal Agassiz systems in [2].

From now on it will be always assumed that K, I are non-empty classes of algebras of type τ, ϱ respectively and $N : \mathbf{P}(\tau) \to \mathbf{P}(\varrho)$ is a naming functor such that $I \subseteq WSC(N)$.

THEOREM 1.

(i) $lim[I, K, N] = lim(I, K, \overline{N})$,

(ii) $Sm(Id_{\overline{N}}(I) \cap Id(K)) \subseteq Id(lim[I, K, N]) \subseteq Id_{\overline{N}}(I) \cap Id(K)$.

THEOREM 2. *If* $I \not\subseteq SC(N)$ *then every identity in* $Id_{\overline{N}}(I)$ *is symmetric and* $Id(lim[I, K, N]) = Id_{\overline{N}}(I) \cap Id(K)$.

REMARK. If $I \subseteq SC(N)$ then by Lemma 3 (iii) it follows that $Id_{\overline{N}}(I) = Id_N(I)$. If $I \not\subseteq SC(N)$ then still $Id_{\overline{N}}(I) \subseteq Id_N(I)$ by Lemma 3 (ii) but the converse inclusion does not hold in general. A suitable example shows that even symmetric identities of $Id_N(I)$ nee not belong to $Id_{\overline{N}}(I)$.

References

[1] G. Grätzer, J. Sichler, *Agassiz sum of algebras*, **Colloquium Mathematicum** 30 (1974), pp. 57–59.

[2] E. Graczyńska, A. Wroński, *On normal Agassiz systems of algebras*, this volume.

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