Bulletin of the **S**ection of **L**ogic
Volume 4/4 (1975), pp. 154–160

reedition 2012 [original edition, pp. 154–162]

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ON ALGEBRA CONNECTED WITH NOTION OF SATISFIABILITY IN THEORIES WITH CONDITIONAL DEFINITIONS

Let T_D be an arbitrary mathematical theory based on first-order calculus of quantifiers with identity and functions and including conditional definitions of functional symbols.

The problem of construction of definitions and sensible expressions has been thoroughly discussed in [1]. In this present paper we shall investigate only expressions with one free variable x. Let a set of such expressions be denoted by S and let V stand for the domain of x. Expressions of the set S are of the form

$$\Psi^A \Rightarrow A \tag{1}$$

The antecedent Ψ^A is called an expression ensuring sensibility of A.

As a consequence of studies included in [1] we obtain

Remark 1. For a given value of the free variable x the expression Ψ^A being the antecedent of implication (1) is false if and only if A has no sense for this value.

Here are some examples of sensible expressions of arithmetics of real numbers:

$$x \neq 0 \Rightarrow \frac{1}{x} > 0 \tag{2}$$

$$x \neq 0 \wedge x + 1 > 0 \Rightarrow \frac{1}{x} + ln(x+1) > ln(x+1) \tag{3}$$

Apart from sensible expressions we consider also sentential expressions (formulas), i.e. expressions built of symbols of the theory T_D in accordance with the theory of syntactic categories. Obviously, each sensible expression is a sentential expression. We also assume that any sentential expression without conditionally defined terms is a sensible expression.

The notion of satisfiability for sentential expressions without conditionally defined terms is identical with that usually accepted. We say that expression (1) is satisfied by $a \in V$ if and only if a satisfies both the antecedent and the consequent of (1).

In our study we shall use the abstraction symbol $\{x \in V : A\}$, where A is an arbitrary sentential expression. From properties of expression (1) given in [1] we have

Remark 2. All properties of the abstraction symbol which hold in the set theory remain valid in the theory T_D. The symbols $\cup, \cap$, will denote normal set-theoretical operations on subsets of V.
Moreover we assume that

$a \in \{x \in V : A\}$ if and only if a satisfies A.

Hence and from Remark 1 we obtain the expression

$$\{x \in V : A\} \subseteq \{x \in V : \Psi^A\}. \tag{4}$$

In case when A is an expression without conditionally defined terms we assume that $\{x \in V : \Psi^A\} = V$.

It follows from Remark 2 and (4) that

$$\{x \in V : A\} = \{x \in V : \Psi^A\} \cap \{x \in V : A\}. \tag{5}$$

In [1] certain operations on sensible expressions have been defined as follows:

$$\ulcorner \Psi^A \Rightarrow A \urcorner \mathrel{\overset{\vee}{\vee}} \ulcorner \Psi^B \Rightarrow B \urcorner = \ulcorner \Psi^A \wedge \Psi^B \Rightarrow A \vee B \urcorner \tag{6}$$

$$\ulcorner \Psi^A \Rightarrow A \urcorner \mathrel{\overset{\wedge}{\wedge}} \ulcorner \Psi^B \Rightarrow B \urcorner = \ulcorner \Psi^A \wedge \Psi^B \Rightarrow A \wedge B \urcorner \tag{7}$$

$$\approx \ulcorner \Psi^A \Rightarrow A \urcorner = \ulcorner \Psi^A \Rightarrow \sim A \urcorner, \tag{8}$$

where $\ulcorner \Psi^A \Rightarrow A \urcorner$, $\ulcorner \Psi^B \Rightarrow B \urcorner$ are arbitrary sensible expressions of T_D and $\wedge, \vee, \sim$ are functors of the classical sentential calculus.

Notice that if $\ulcorner \Psi^A \Rightarrow A \urcorner$, $\ulcorner \Psi^B \Rightarrow B \urcorner \in S$, then the results of operations $\overset{\vee}{\vee}$, $\overset{\wedge}{\wedge}$, $\approx$ on those expressions also belong to S, as formulas with one free variable x. Thus we have the following algebra of formulas:

$$\mathcal{A} = (S; \overset{\vee}{\vee}, \overset{\wedge}{\wedge}, \approx)$$

in which operations are defined by (6) – (8).

Consider a mapping

$$f : S \to 2^V \times 2^V$$

defined for an arbitrary formula $\ulcorner \Psi^A \Rightarrow A \urcorner \in S$ as follows:

$$f(\Psi^A \Rightarrow A) = \langle \{x \in V : \Psi^A\}, \{x \in V : \Psi^A\} \cap \{x \in V : A\} \rangle.$$

With the use of (5), the definition of f can be formulated in a simpler way, for our aims, however, it is of no need.

It can easily be seen that f generates in S a relation E_f of the equivalence type defined by the following formula:

$\ulcorner \Psi^A \Rightarrow A \urcorner E_f \ulcorner \Psi^B \Rightarrow B \urcorner$ if and only of
$f(\Psi^A \Rightarrow A) = f(\Psi^B \Rightarrow B)$ for arbitrary formulas
$\ulcorner \Psi^A \Rightarrow A \urcorner, \ulcorner \Psi^B \Rightarrow B \urcorner \in S$

The set $\{x \in V : \Psi^A\}$ will be called a region of sensibility of A and the set $\{x \in V : \Psi^A\} \cap \{x \in V : A\}$ will be region of truthfulness of A in V. Thus, two expressions of the set S are equivalent in the sense of a relation E_f if and only if they have the same regions of sensibility and truthfulness in V. For instance, expressions (2) and (3) are not equivalent in the set of real numbers due to the fact that their regions of sensibility are different, although they have the same region of truthfulness.

LEMMA 1. *Relation E_f is a congruence in the algebra $\mathcal{A} = (S; \overset{\vee}{\vee}, \overset{\wedge}{\wedge}, \approx)$.*

Consider now a set $f(S)$. It is a subset of the set $2^V \times 2^V$ with the following property:

If $X, Y \in 2^V$ and $\langle X, Y \rangle \in f(S)$, then $Y \subseteq X$.

This property is an immediate consequence of the definition of f and (4).

Assume now that for each pair $\langle X, Y \rangle \in 2^V \times 2^V$ such that $Y \subseteq X$ there exists an expression $\ulcorner \Psi^A \Rightarrow A \urcorner \in S$ satisfying $f(\Psi^A \Rightarrow A) = \langle X, Y \rangle$.

Thus we have

$$f(S) = \{\langle X, Y\rangle \in 2^V : X \supseteq Y\}. \tag{9}$$

In the set $f(S)$ the function f generates operations $+, \cdot, \neg$ as follows.

$$f(\Psi^A \Rightarrow A) = f(\Psi^B \Rightarrow B) = f(\ulcorner \Psi^A \Rightarrow A\urcorner \overset{\vee}{\vee} \ulcorner \Psi^B \Rightarrow B\urcorner) \tag{10}$$

$$f(\Psi^A \Rightarrow A) \cdot f(\Psi^B \Rightarrow B) = f(\ulcorner \Psi^A \Rightarrow A\urcorner \overset{\wedge}{\wedge} \ulcorner \Psi^B \Rightarrow B\urcorner) \tag{11}$$

$$\neg f(\Psi^A \Rightarrow A) = f(\approx \ulcorner \Psi^A \Rightarrow A\urcorner). \tag{12}$$

Let us introduce the following denotations:

$\{x \in V : \Psi^A\} = X_1$
$\{x \in V : \Psi^B\} = X_2$
$\{x \in V : A\} = Y_1$
$\{x \in V : B\} = Y_2.$

By the use of these denotations, by Remark 2, (5) and the definition of f, equalities (10), (11) and (12) may be rewritten as:

$$\langle X_1, Y_1\rangle + \langle X_2, Y_2\rangle = \langle X_1 \cap X_2, Y_1 \cap X_2 \cup Y_2 \cap X_1\rangle \tag{10$'$}$$

$$\langle X_1, Y_1\rangle \cdot \langle X_2, Y_2\rangle = \langle X_1 \cap X_2, Y_1 \cap Y_2\rangle \tag{11$'$}$$

$$\neg\langle X_1, Y_1\rangle = \langle X_1, X_1 \cap Y_1'\rangle, \tag{12$'$}$$

respectively.

Hence we can consider an algebra

$$\mathcal{B} = (f(S); +, \cdot, \neg)$$

in which operations are defined by $(10') - (12')$. Notice that these operations are closely connected with operations defined by $D2 - D4$ in [1].

LEMMA 2. *An algebra* $\mathcal{B} = (f(S); +, \cdot, \neg)$ *is a homomorphic image of the algebra* $\mathcal{A} = (S; \overset{\vee}{\vee}, \overset{\wedge}{\wedge}, \approx)$.

THEOREM 1. *The algebra* $\mathcal{B}$ *is isomorphic with the algebra* $\mathcal{A}/E_f$.

Consider the following relation in $\mathcal{B}$:

$$x \sim y \text{ if and only if } (\neg x)x = (\neg y)y.$$

With the aid of (9), (10′) – (12′) it can easily be verified that the relation $\sim$ is a congruence in $\mathcal{B}$. In accordance with this relation pairs $\langle X_1, Y_1\rangle, \langle X_2, Y_2\rangle \in f(S)$ are equivalent if and only if their first elements are equal, i.e. $X_1 = X_2$.

THEOREM 2. *If $\langle X, Y\rangle \in f(S)$, then the class $[\langle X, Y\rangle]_{\sim, f(S)}$ is a Boolean algebra.*

Let us write

$$\langle X, Y\rangle_{\sim, f(S)} = A_X$$

and

$$\mathcal{A}_X = (A_X; +, \cdot, \neg).$$

In the set 2^V we define a relation $\leqslant$ as follows:

$$X \leqslant Y \text{ if and only if } Y \subseteq X.$$

Obviously, the system $(2^V, \leqslant)$ is a partially ordered set with the supremum property, and for arbitrary $X, Y \in 2^V$ we have

$$sup\{X, Y\} = X \cap Y.$$

For any $X, Y \in 2^V$ such that $X \leqslant Y$ we define a mapping

$$h_X^Y : A_X \to A_Y$$

in the following way:

$$h_X^Y(\langle X, Y_1\rangle) = \langle Y, Y_1 \cap Y\rangle \text{ for } \langle X, Y_1\rangle \in A_X. \tag{13}$$

THEOREM 3. *If $X, Y \in 2^V$ and $X \leqslant Y$, then h_X^Y is a homomorphism of the algebra $\mathcal{A}_X$ into the algebra $\mathcal{A}_Y$ satisfying the following conditions:*

(a) $h_Y^Z \cdot h_X^Y = h_X^Z$ for $X \leqslant Y \leqslant Z$

(b) h_X^X is an identity function on A_X.

THEOREM 4. *A system $L = \langle 2^V, \{\mathcal{A}_X\}_{X \in 2^V}, \{h_X^Y\}_{X,Y \in 2^V, X \leqslant Y}\rangle$ is a simple system of Boolean algebras and such that if $X, Y \in 2^V$ and $X \neq Y$, then $A_X \cap A_Y = \emptyset$.*

It follows from this theorem that there exists an algebra

$$\mathcal{A} = (\bigcup_{X \in 2^V} A_X; \#, \circ, \neg\!\!\neg)$$

which is a union of the simple system L and in which operations are defined as follows.

If $x, y \in \bigcup_{X \in 2^V} A_X$ and $x \in A_X, y \in A_Y$, then

$$x \# y = h_X^{X \cap Y}(x) + h_Y^{X \cap Y}(y) \tag{14}$$

$$x \circ y = h_X^{X \cap Y}(x) \cdot h_Y^{X \cap Y}(y) \tag{15}$$

$$\neg\!\!\neg x = \neg h_X^{X \cap Y}(x) \tag{16}$$

THEOREM 5. *An algebra $\mathcal{B} = (f(S), +, \cdot, \neg)$ is a union of a single system of Boolean algebras.*

Let us now denote some thought to interpretation of the results obtained so far. It is clear that if we allow occurrence of terms which lose their sense in formulas, then sets determined by these formulas do not satisfy laws of a Boolean algebra.

Consider sets determined by the following sentential formulas in the set of real numbers R,

$$\frac{1}{x} > 0 \tag{17}$$

$$\frac{1}{x+1} > 0 \tag{18}$$

These sets are $\{x \in R : \frac{1}{x} > 0\} = (0, \infty)$ and $\{x \in R : \frac{1}{x+1} > 0\} = (-1, \infty)$. Performing operations on the sets $(0, \infty)$, $(-1, \infty)$ we usually forget about sentential forms which determined these sets and, for instance, we calculate

$$(0, \infty) = (-\infty, 0)$$

$$(0, \infty)' \cup (-1, \infty) = (-1, \infty),$$

while sets determined by negation of formula (17) and disjunction of forms (17) and (18) are $(-\infty, 0)$ and $(-1, 0) \cup (0, \infty)$, respectively.

Algebra $\mathcal{B}$ constructed in the present allows us to perform operations on sets determined by sentential forms which lose their sense in the way is usually done in mathematics. In the case of formulas (17) and (18), the following sensible expressions should be considered:

(17′) $\quad x \neq 0 == \frac{1}{x} > 0$

(18′) $\quad x + 1 \neq == \frac{1}{x+1} > 0$

Formulas (17′) and (18′) determine the pairs of sets

$$\langle(-\infty, 0) \cup (0, \infty), (0, \infty)\rangle$$

$$\langle(-\infty, -1) \cup (-1, \infty), (-1, \infty)\rangle,$$

respectively.

Performing the suitable operations on these pairs in algebra $\mathcal{B}$ we obtain

$$\neg\langle(-\infty, 0) \cup (0, \infty), (0, \infty)\rangle = \langle(-\infty, 0) \cup (0, \infty), (-\infty, 0)\rangle$$

$$\langle(-\infty, 0) \cup (0, \infty), (0, \infty)\rangle + \langle(-\infty, -1) \cup (-1, \infty), (-1, \infty)\rangle =$$

$$= \langle(-\infty, -1) \cup (-1, 0) \cup (0, \infty), (-1, 0) \cup (0, \infty)\rangle.$$

Second elements of the pairs obtained in that way are the sets determined by negation of formula (17) and disjunction of formulas (17) and (18).

References

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