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THE DEGREES OF COMPLETENESS OF DUAL COUNTERPARTS OF ŁUKASIEWICZ SENTENTIAL CALCULI

1. In this paper we use the terminology of [1]. Moreover, let $Sb(X) = \{e\alpha | \alpha \in X\}$ and e is any substitution in $\mathcal{L}$, $\overline{dC_k}(X) = dC_k(Sb(X))$ and $d\text{Ł}_k = taut(\overline{\mathcal{M}}_k)$.

2. The notions of the ordinal and cardinal degree of completeness of a consequence operation has been introduced by A. Tarski in [2] and [3] respectively. We shall give modified definitions of these notions.

Definition 1. The ordinal degree of completeness of C, in symbol $\overline{g}(C)$, is the smallest ordinal number $\mu \neq 0$ which satisfies the following condition: there exists no increasing sequence of type μ of consistent invariant C-systems which begins with $C(\emptyset)$.

Definition 2. The cardinal degree of completeness of C, in symbols $g(C)$, is the number of invariant C-systems.

In [1] there is proved that

(∗) $d\text{Ł}_k \subseteq d\text{Ł}_l$ iff $l - 1$ divides $k - 1$, in symbols $l - 1 | k - 1$.

This fact enables us to count degrees of completeness a consequence operation dC_k by means of the method used by M. Tokarz (cf. [4] and [5]).

Lemma 1. *$dC_k(X) \subseteq dL_l$ if and only if $X \subseteq dL_l$ and $dL_k \subseteq dL_l$.*

Proof: Implication from left to right is obvious. Suppose that $\alpha \notin d\text{Ł}_l$, i.e., there exists a valuation $h_1 : \mathcal{L} \to \overline{\mathcal{M}}_l$ such that $h_1(\alpha) = 1$. On the other hand observe that $Sb(X) \subseteq d\text{Ł}_l$. Hence for every $h : \mathcal{L} \to \overline{\mathcal{M}}_l$,

$h(Sb(X)) \subseteq A_l - \{1\}$. Consequently, $h_1(Sb(X)) \subseteq A_l - \{1\}$. Since $d\text{Ł}_k \subseteq d\text{Ł}_l$ then $l-1|k-1$. Therefore h_1 is a valuation in $\overline{\mathcal{M}}_k$. Hence $\alpha \notin \overline{dC}_k(X)$. This complete the proof of Lemma 1.

THEOREM 1. *If* $dL_k = \bigcap\{dL_i | i \in I\}$ *then* $k \in I$.

PROOF: Notice that I is finite. Assume that $k \notin I$. Let $a = lcm(i-1|i \in I \cup \{k\})$, where lcm being the least common multipla. Let $\alpha = \alpha(p)$ and

$$\alpha(\frac{X}{a}) \begin{cases} < 1 & \text{if there exists } j \in I \text{ such that } a_j | x \\ & \text{where } a_j = \frac{a}{j-1} \\ = 1 & \text{otherwise.} \end{cases}$$

Observe that $\alpha \in \bigcap\{d\text{Ł}_i i \in I\}$ but $\alpha \notin d\text{Ł}_k$. Therefore $k \in I$.

THEOREM 2. $\overline{dC}_n(X) = \bigcap\{dL_i | \overline{dC}_n(X) \subseteq dL_i\}$.

PROOF: Inclusion $\subseteq$ is obvious. In view of Lemma 1 it is sufficient to prove that $\bigcap\{d\text{Ł}_i | X \subseteq d\text{Ł}_i \text{ and } d\text{Ł}_n \subseteq d\text{Ł}_i\} \subseteq \overline{dC}_n(X)$.

I. Let $X = \{\alpha\}$, where $\alpha = \alpha(p_1, \ldots, p_n)$. Moreover let $d\text{Ł}_{k_1}, \ldots, d\text{Ł}_{k_l}$ and $d\text{Ł}_{j_1}, \ldots, d\text{Ł}_{j_m}$ be sequences of all d-tautologies satisfying the following conditions:

(1) $\alpha \in d\text{Ł}_{k_1} \cap \ldots \cap d\text{Ł}_{k_l}$ and $d\text{Ł}_n \subseteq d\text{Ł}_{k_1} \cap \ldots \cap d\text{Ł}_{k_l}$,
(2) $\alpha \notin d\text{Ł}_{j_i}$, for $i = 1, \ldots, m$ and $d\text{Ł}_n \subseteq d\text{Ł}_{j_1} \cap \ldots \cap d\text{Ł}_{j_m}$.

In virtute of $(*)$ we have that these sequences are finite.

Observe that if there exists no sequences are finite condition (2) then $\alpha \in d\text{Ł}_n$, so the theorem is evident.

In view of condition (2) the formula α is demolished in any matrix $\overline{\mathcal{M}}_{j_i}$ for $i = 1, \ldots, m$. Hence there exists sequences $\frac{x_1^i}{n-1}, \ldots, \frac{x_s^i}{n-1} \in A_{j_i}$, for $i = 1, \ldots, m$, such that $\overline{\alpha}(\frac{x_1^i}{n-1}, \ldots, \frac{x_s^i}{n-1}) = 1$. Suppose that $\beta \in \bigcap\{d\text{Ł}_i \; i = 1, \ldots, l\}$. We shall show that $\beta \in \overline{dC}_n(\{\alpha\})$. Due to the fact that $dC_n = C^*_{\overline{n}}$ (see [1]) it suffice to prove that for every valuation $h : \mathcal{L} \to \overline{\mathcal{M}}_n$, if $h(Sb(\{\alpha\})) \subseteq A_n - \{1\}$ then $h(\beta) < 1$.

Let $h_1 : \mathcal{L} \to \overline{\mathcal{M}}_n$ and let $h_1(Sb(\{\alpha\})) \subseteq A_n - \{1\}$. Notice that $h_1(V)$, where V is the set of all propositional variables, is a finite set. Let us put $h_1(V) = \{\frac{y_{r_1}}{n-1}, \ldots, \frac{y_{r_k}}{n-1}\}$. Consider two cases:

a) $gcd(y_{r_1}, \ldots, y_{r_k}, n-1) = 1$,

b) $gcd(y_{r_1}, \ldots, y_{r_k}, n-1) \neq 1$,

gcd being the greatest common divisor.

Ad a). Let $\varphi_1, \ldots, \varphi_s$ be arbitrary formulas satisfying the following equalities: $h_1(\varphi_1) = \frac{x_1^1}{n-1}, \ldots, h_1(\varphi_s) = \frac{x_s^1}{n-1}$. It follows from the McNaughton's criterion that such formulas do exist. Therefore $h_1(\alpha(\varphi_1, \ldots, \varphi_s)) = 1$ Hence $h_1(Sb(\{\alpha\})) \not\subseteq A_n - \{1\}$ contrary to our assumption.

Ad b). Let $gcd(y_{r_1}, \ldots, y_{r_s}, n-1) = d > 1$. Put $c = \frac{n-1}{d}$. Notice that $\frac{y_{r_1}}{n-1}, \ldots, \frac{y_{r_k}}{n-1} \in A_{c+1}$. There are two possibilities:

b$_1$) $c+1 \in \{j_1, \ldots, j_m\}$,

or else

b$_2$) $c+1 \notin \{j_1, \ldots, j_m\}$.

Ad b$_1$). Let for example $c+1 = j_p$ for a certain $p \in \{1, \ldots, m\}$. On the grounds of an argument similar to that in a) we have that there exist formulas $\Psi_1, \ldots, \Psi_s$ such that $h_1(\Psi_1) = \frac{x_1^p}{n-1}, \ldots, h_1(\Psi_s) = \frac{x_s^p}{n-1}$. Hence $h_1(\alpha(\Psi_1, \ldots, \Psi_s)) = 1$ and then $h_1(Sb(\{\alpha\})) \not\subseteq A_n - \{1\}$ contrary to our assumption.

Ad b$_2$). Since c divides $n-1$ then $d\text{Ł}_n \subseteq d\text{Ł}_{c+1}$. Hence $c+1 \in \{k_1, \ldots, k_l\}$. Let $c+1 = k_p$ for a certain $p \in \{1, \ldots, l\}$. Therefore $h_1(V) \subseteq A_{k_p}$. Since $\beta \in \bigcap\{d\text{Ł}_i | i \in \{1, \ldots, l\}\}$ then $\beta \in d\text{Ł}_{k_p}$. This means that $h_1(\beta) < 1$, and the proof is complete for $x = \{\alpha\}$.

II. Assume now that X is a set of formulas. Let $d\text{Ł}_{k_1}, \ldots, d\text{Ł}_{k_l}$ and $d\text{Ł}_{j_1}, \ldots, d\text{Ł}_{j_m}$ be sequences of all the systems of d-tautologies satisfying the following conditions:

(1′) $X \subseteq d\text{Ł}_{k_1} \cap \ldots \cap d\text{Ł}_{k_l}$ and $d\text{Ł}_n \subseteq d\text{Ł}_{k_1} \cap \ldots \cap d\text{Ł}_{k_l}$,
(2′) $X \not\subseteq d\text{Ł}_{j_i}$ for $i = 1, \ldots, m$ and $d\text{Ł}_n \subseteq d\text{Ł}_{j_1} \cap \ldots \cap d\text{Ł}_{j_m}$,

In virtue of (2′) there exist formulas $\alpha_{j_1}, \ldots, \alpha_{j_m}$ such that $\alpha_{j_i} \in X$ and $\alpha_{j_i} \notin d\text{Ł}_{j_i}, \alpha_{j_i} \notin d\text{Ł}_{i_i}$ for $i = 1, \ldots, m$. Let $\alpha = \alpha_{j_1} \vee \ldots \vee \alpha_{j_m}$. Notice that $\alpha \in d\text{Ł}_{k_1} \cap \ldots \cap d\text{Ł}_{k_l}$ and $\alpha \notin d\text{Ł}_{j_1}$ for $i = 1, \ldots, m$. From (I) we have already known that $\overline{dC}_n(\{\alpha\}) = \bigcap\{d\text{Ł}_{k_i} | i = 1, \ldots, l\}$. Since $dC_n(\{\alpha\}) \subseteq dC_n(X)$ (see Lemma 5.1(iii) and Theorem 4.1b) in [1]) then $\overline{dC}_n(\{\alpha\}) \subseteq \overline{dC}_n(X)$. Hence $\bigcap\{d\text{Ł}_{k_i} | i = 1, \ldots, l\} \subseteq \overline{dC}_n(X)$. This completes the proof.

THEOREM 3. *$\overline{g}(dC_n) = p(n-1)+1$, where $p(m)$ is the number of divisors of m (including m and 1).*

PROOF: Let $d_1, \ldots, d_{p(n-1)}$ be the decreasing sequence of all natural numbers d_j such that d_{j-1} divides $n-1$. Let $I_j = \{d_j, d_{j+1}, \ldots, d_{p(n-1)}\}$. In virtute of Theorem 1 we have that $d\text{Ł}_n = \bigcap\{d\text{Ł}_i | i \in I_1\} \not\subseteq \bigcap\{d\text{Ł}_i | i \in I_2\} \not\subseteq \ldots \not\subseteq \bigcap\{d\text{Ł}_i | i \in I_{p(n-1)}\} \not\subseteq L$. Hence $\overline{g}(dC_n) \geqslant p(n-1)+1$.

In virtute of Theorem 2 we have that $\overline{g}(dC_n) \leqslant p(n-1)+1$. This complete the proof.

Conclusion: $\overline{g}(C_n) = \overline{g}(dC_n)$.

LEMMA 2. *If $k \geqslant m$ then $\urcorner_k(((p \to_n g) \to p) \to p) \in d\text{Ł}_m$ iff $n \geqslant m-1$.*

PROOF: Suppose that $n < m-1$. Let h_1 be a valuation such that $h_1(p) = \frac{m-2}{m-1}$ and $h_1(q) = 0$. It is easy to see that $h_1(\urcorner_k(((p \to_n g) \to p) \to p)) = 1$. Let now $n \geqslant n-1$. Then for any valuation h in the matrix $\overline{\mathcal{M}}_m$, $h(((p \to_n g) \to p) \to p) = 1$. Hence $h(\urcorner_k(((p \to_n g) \to p) \to p)) = 0$. Therefore this formula belongs to $d\text{Ł}_m$.

Let $D_1 = d\text{Ł}_{k_1} \cap \ldots \cap d\text{Ł}_{k_l}$ and $D_2 = d\text{Ł}_{m_1} \cap \ldots \cap d\text{Ł}_{m_n}$. Moreover let the following conditions be satisfied:

(i) If $a < b$ then $k_a > k_b$ and $m_a > m_b$.
(ii) If $k_a > k_b$ ($m_a > m_b$) then $d\text{Ł}_{k_a} \not\subseteq d\text{Ł}_{k_b}$ ($d\text{Ł}_{m_a} \not\subseteq d\text{Ł}_{m_b}$).

LEMMA 3. *If $D_1 = D_2$ then $l = n$ and $k_i = m_i$ for every i, $1 \leqslant i \leqslant n$.*

PROOF: Put $k > k_i$ and $k > m_j$ for every $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, n\}$. Assume that $m_1 \neq k_1$. For example let $m_1 > k_1$. In virtute of Lemma 2, $\urcorner_k(((p \to_{k_1-1} g) \to p) \to p) \in D_1$. However $k-1 < m_1 - 1$. Therefore $\urcorner_k(((p \to_{k_1-1} g) \to p) \to p) \notin d\text{Ł}_{m_1}$. Hence this formula does not belong to D_2. This fact implies that $D_1 \neq D_2$ contrary to our assumption. Hence $m_1 = k_1$.

Now assume that b is the smallest number such that $k_b \neq m_b$. For example let $m_b < k_b$. Put $a = lcm(k_1-1, \ldots, k_l-1, m_1-1, \ldots, m_n-1)$ and $k_1^a = \frac{a}{k_1-1}, \ldots, m_n^a = \frac{a}{m_n-1}$. Let $\alpha = \alpha(p)$ be a formula defined in the matrix $\overline{\mathcal{M}}_{a+1}$ in the following way:

$$(\cdot)\ \overline{\alpha}(\tfrac{x}{a}) \begin{cases} < 1 & \text{if } m_1^a | x \text{ or } \ldots \text{ or } m_n^a | x, \\ = 1 & \text{otherwise.} \end{cases}$$

It is easy to see that $\alpha \in D_2$. We shall show that $\alpha \notin D_1$.

Suppose that $\alpha \in d\text{Ł}_{k_b}$. Hence $\overline{\alpha}(\frac{1}{k_b - 1}) = \overline{\alpha}(\frac{k_b^a}{a} < 1$. By $(\cdot)$ we have that there exists an i, $1 \leqslant i \leqslant n$, such that $m_i^a | k_b^a$. Therefore $k_b - 1 | m_i - 1$. Since $i \geqslant b$ then $m_i \leqslant m_b$. Since $m_b < k_b$ then $m_i < k_b$, so $k_b - 1$ does not divide $m_i - 1$. Hence $i < b$. But then $m_i = k_i$. Therefore $k_b - 1 | k_i - 1$. This implies that $d\text{Ł}_{k_i} \subseteq d\text{Ł}_{k_b}$ contrary to our construction of D_1. So we obtain $\alpha \notin D_1$. Hence $D_1 \neq D_2$. This finishes the proof of Lemma 3.

Theorem 4. $g(dC_n) = g(C_n) = \sum_{a_1,\ldots,a_k} P(a_i) + 1$, *where the sequence* $\{a_1, \ldots, a_k\}$ *has the following properties:* $a_1 = n$; $a_1 > a_2 > \ldots > a_k > 1$; *for every* $i \in \{1, \ldots, k\}$ $a_i - 1 | n - 1$ *and* $P(a_i)$ *is the number of all subsequences.* Sc *of* $\{a_1, \ldots, a_k\}$ *such that:* $a_i \in Sc$; *for any* $b \in Sc$, $a_i \geqslant b$; *if* $a_j, a_k \in Sc$ *and* $a_j < a_k$ *then* $a_j - a \not| a_k - 1$.

This theorem follows immediately from Lemma 3 and Theorem 2.

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