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Bernd I. Dahn

NEIGHBOURHOOD SEMANTICS AND GENERALIZED KRIPKE SEMANTICS

It is proved in [2] that generalized Kripke semantics has the same depth as the semantics of propositional matrices with exactly one designated element and hence for modal logics has greater depth than the Boolean semantics (cf. [3] for the concept of depth). This result has been considerably extended by K. Bernhardt [1]. The aim of this note is to prove that generalized Kripke semantics for modal propositional logic with standard interpretation of classical propositional connectives has the same depth as the neighbourhood semantics.

A generalized Kripke frame with standard interpretation of classical propositional connectives ($GKMSt$-frame for short) is a pair $(\mathcal{A}, H)$ where $\mathcal{A}$ is some algebraic system and H is a formula of the first order language built from symbols of the signature of $\mathcal{A}$ and one new unary relational symbol P and such that H has exactly the individual variable x_0 free. Let A denote the universe of $\mathcal{A}$. By Var and $\underline{P}$ we denote the set of propositional variables and the powerset operator, respectively. A *model over* $(\mathcal{A}, H)$ is given by an assignment $V : Var \to \underline{P}(A)$. We define a relation $\Vdash_V$ between elements of A and modal formulas as follows.

1) If $p \in Var$ then $a \Vdash_V p$ iff $a \in V(p)$
2) For all α, $a \Vdash_V \neg\alpha$ iff not $a \Vdash_V \alpha$
3) For all α, β, $a \Vdash_V \alpha \wedge \beta$ iff $a \Vdash_V \alpha$ and $a \Vdash_V \beta$
4) For all α, $a \Vdash_V \Box\alpha$ iff $(\mathcal{A}, P, a) \models \mathsf{H(a)}$ where $P = \{b \in A : b \Vdash_V \alpha\}$.

α is *true in the model* V if $a \Vdash_V \alpha$ for every $a \in A$. A *neighbourhood frame* is a pair (U, N) where U is a set and N is a function, $N : U \to \underline{P}(\underline{P}(U))$.

A *model over* (U, N) is given by an assignment $V : Var \to \underline{P}(U)$. Define a relation $\vdash_V$ by 1), 2), 3) as above with $\Vdash_V$ replaced by $\vdash_V$ and

4′) For all α, $a \vdash_V \Box\alpha$ iff $\{b \in U : b \vdash_V \alpha\} \in N(a)$.

Again, α is *true in this model* if $U \vdash_V \alpha$ for every $u \in U$. A model V over a frame F (which may be either a $GKMSt$-frame or a neighbourhood frame) is said to be a *model* of a set Σ of modal formulas if every $\sigma \in \Sigma$ is true in this model. It is said to be a model of a single formula α if it is a model of $\{\alpha\}$. The *logic defined by the frame* F is the binary consequence relation $\models$ between sets of modal formulas and single formulas, induced by F, i.e. $\Sigma \models \alpha$ iff every model of Σ over F is a model of α. Frames are said to be *equivalent* if they define the same logic.

THEOREM 1. *For every GKMSt-frame* $(\mathcal{A}, H)$ *there is an equivalent neighbourhood frame* (U, N)*. If* $\mathcal{A}$ *is a finite algebraic system, then* U *can be chosen as a finite set.*

PROOF. Let U be the universe of $\mathcal{A}$. Define the function N over U by

$$X \in N(u)\ iff\ (\mathcal{A}, P, u) \models \mathsf{H(u)} \text{ where } \mathsf{P} = \mathsf{X}.$$

It is an easy induction to check that for every $u \in U$, for every model V and for every α

$$u \Vdash_V \alpha \text{ iff } u \vdash_V \alpha.$$

Hence $(\mathcal{A}, H)$ and (U, N) define the same logic. Q.E.D.

THEOREM 2. *For every neighbourhood frame* (U, N) *there is an equivalent GKMSt-frame* $(\mathcal{A}, H)$*. If* U *is a finite set, then* $\mathcal{A}$ *can be chosen as a finite algebraic system.*

PROOF. Let (U, N) be any given neighbourhood frame. By k we denote the least cardinal not less than $card\underline{P}(U)$ and such that $k = \lambda$, $cardU$ for same cardinal λ. Let A be any set of power k and let f be an embedding of $\underline{P}(U)$ into A. If U is infinite we have $k = card\underline{P}(U)$ and we may take $A = \underline{P}(U)$. In this case f may be as chosen the identity on $\underline{P}(U)$ and the following proof is slightly simplified. R denotes the range of the function f. In order to simplify our notation, we shall write $f(u)$ instead of $f(\{u\})$ for $u \in U$. Similarly, if $f^{-1}(a)$ is defined and is a singleton, it will be identified with it's element.

Let E be an equivalence relation over A such that $\{f(u) : u \in U\}$ is just one equivalence class of E and such that each equivalence class of E is of power $cardU$. E exists by the choice of k. Define $a \leqslant b$ iff $a, b \in R$ and $f^{-1}(a) \subseteq f^{-1}(b)$. Let the function $h : A \to A$ be such that the restriction of h to any class of E maps this class one-one onto $\{f(u) : u \in U\}$ (see fig. 1).

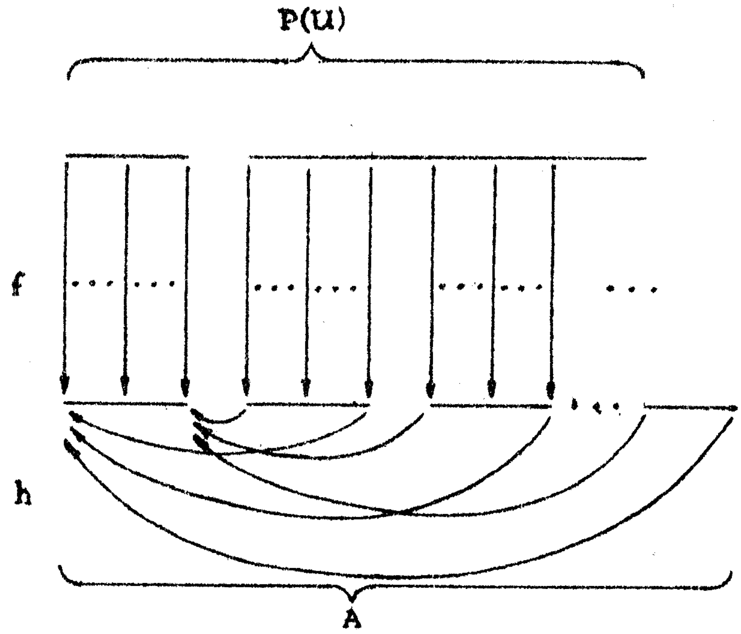

Fig. 1.

The binary relation r over A is defined by

$$r(a,b) \text{ iff } b \in R \text{ and } f^{-1}(b) \in N(f^{-1}h(a)).$$

Now we are ready to define

$$\mathcal{A} = (A, R, E, \leqslant, h, r).$$

$At(x)$ is an abbreviation of the formula

$$R(x) \wedge \exists y(y \leqslant x \wedge \neg y = x \wedge \forall z(z \leqslant x \rightarrow z = x \vee z = y)).$$

Obviously, At defines $\{f(u)|u \in U\}$ in $\mathcal{A}$. Now let H be the formula

$$\forall x_1(R(x_1) \rightarrow (\forall x_2(At(x_2) \rightarrow (x_2 \leqslant x_1 \rightarrow \exists x_3(E(x_0, x_3) \ \wedge \ P(x_3) \wedge$$

$$x_2 = h(x_3))))) \rightarrow r(x_0, x_1)).$$

With every modal V over $(\mathcal{A}, H)$ and with every $a \in A$ we associate as assignment $V_a : Var \rightarrow \underline{P}(U)$ by

$$V_a(p) = \{f^{-1}h(b) : E(a,b) \text{ and } b \in V(p)\}.$$

CLAIM. For every $a, b \in A$ such that $E(a,b)$, for every $V : Var \rightarrow \underline{P}(A)$ and for every modal propositional formula

$$b \Vdash_V \alpha \text{ iff } f^{-1}h(b) \vdash_{V_a} \alpha.$$

The claim is proved by induction on the form of α. We demonstrate the only nontrivial step.

Assume $b \Vdash_V \Box\alpha$. Then $(\mathcal{A}, P, b) \models H(b)$ with $P = \{c : c \Vdash_V \alpha\}$ and hence we infer that $r(b, x_1)$ for

$$x_1 = f(\{f^{-1}h(x_3) : E(b, x_3) \text{ and } x_3 \Vdash_V \alpha\}).$$

But $r(b, x_1)$ yields $\{f^{-1}h(x_3) : E(b, x_3)$ and $x_3 \Vdash_V \alpha\} \in N(f^{-1}h(b))$, i.e. $\{u : u \vdash_{V_a} \alpha\} \in N(f^{-1}h(b))$ any our induction hypothesis. Hence $f^{-1}h(b) \vdash_{V_a} \Box\alpha$.

In order to prove the converse, assume that $f^{-1}h(b) \vdash_{V_a} \Box\alpha$, i.e. $\{u : u \vdash_{V_a} \alpha\} \in N(f^{-1}h(b))$. By induction hypothesis we claim that

$$r(b, f(\{u : u \vdash_{V_a} \alpha\})). \tag{1}$$

We have to show that $H(b)$ is true in $(\mathcal{A}, P, b)$ if $P = \{x_3 : x_3 \Vdash_V \alpha\}$. Assume that x_1 satisfies the assumption of $H(b)$. Then for every $u \in U$ $\{u\} \subseteq f^{-1}(x_1)$ iff $u = f^{-1}h(x_3)$ for some $x_3 \in A$ such that $E(b, x_3)$ and $x_3 \Vdash_V \alpha$. Hence

$$\begin{aligned} f^{-1}(x_1) &= \{f^{-1}h(x_3) : E(b, x_3) \text{ and } x_3 \Vdash_V \alpha\} \\ &= \{f^{-1}h(x_3) : E(a, x_3) \text{ and } f^{-1}h(x_3) \vdash_{V_a} \alpha\} \\ &= \{u : u \vdash_{V_a} \alpha\}. \end{aligned}$$

This together with (1) completes the proof of the claim.

By $\models_N$ and $\models_H$ we denote the logics defined by (U, N) and $(\mathcal{A}, H)$ respectively.

Assume that $\Sigma \not\models_H \alpha$. Let V be a model of Σ over $(\mathcal{A}, H)$ such that $a \not\models_V \alpha$ for some $a \in A$. The claim yields immediately that V_a is a model of Σ over (U, N), but $f^{-1}h(a) \not\Vdash_{V_a} \alpha$. Hence $\Sigma \not\models_N \alpha$ therefore

$$\models_N \subseteq \models_H . \tag{2}$$

On the other hand, assume $\Sigma \not\models_N \alpha$ and let V' be ant model of Σ over (U, N) which is not a model of α, say $u \not\Vdash_V \alpha$. Define a model V over $(\mathcal{A}, H)$ by

$$V(p) = \{a : f^{-1}h(a) \in V(p)\}$$

for each $p \in Var$. Observe that $V_a = V'$ for each $a \in A$. By the claim we infer that V is a model of Σ over $(\mathcal{A}, H)$, but $a \not\Vdash_V \alpha$ if $a \in A$ is such that $f^{-1}h(a) = u$. Hence $\Sigma \not\models_H \alpha$ and therefore

$$\models_H \subseteq \models_N \tag{3}$$

(2) and (3) complete the proof of Theorem 2. Q.E.D.

Now the fact that the neighbourhood semantics and the generalized Kripke semantics with standard interpretation of classical propositional connectives have equal depth is an easy corollary of Theorem 1 and 2.

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Sektion Mathematik
Humboldt-Universität zu Berlin, GDR

currently
Department of Mathematics
University of Wrocław